Universidad Autónoma de Madrid

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CONTACT FIBRATIONS OVER THE 2–DISK

Roger Casals

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Supervised by Francisco Presas

Contents

Resultados Principales y Conclusiones

El contenido de esta disertación doctoral es una serie de resultados en el ámbito de la topologia simpléctica y de contacto. Éstas son las geometrías que subyacen la mecánica Hamiltoniana, la termodinámica y la óptica geométrica, siendo establecidas como áreas de investigación actual en matemática pura debido a los trabajos de V.I. Arnol'd, M. Gromov, Y. Eliashberg y los matemáticos de la Escuela Francesa, incluyendo a D. Bennequin, F. Laudenbach y E. Giroux. Es posible distinguir los resultados presentados en esta disertación en tres categorías en función de su relevancia y temática específica.

La primera parte de la tesis, contenida en el Capítulo 2, está dedicada a la resolución de la conjetura de existencia de estructuras de contacto en variedades de casi contacto para variedades 5–dimensionales. Este problema fue propuesto por S.S. Chern en 1966 y se han ido obteniendo soluciones parciales hasta su resolución completa en 2014 por M.S. Borman, Y. Eliashberg y E. Murphy. El caso de variedades abiertas se sigue de la tesis de M. Gromov en 1969, mientras que el resultado para variedades 3–dimensionales fue demostrado por J. Martinet en 1970 y R. Lutz en 1977. Los matemáticos H. Geiges y C.B. Thomas presentaron una solución parcial para el caso 5–dimensional variedades con grupo fundamental controlado. Esta disertación presenta una prueba de la conjetura para toda variedad 5–dimensional, anterior a la resolución completa de la conjetura.

En el Capítulo 3 se hallan los resultados correspondientes a la segunda parte de la tesis. La resolución propuesta por M.S. Borman, Y. Eliashberg y E. Murphy de la conjetura de existencia en topología de contacto introduce una clase distinguida de estructuras de contacto, las estructuras flexibles. Éstas están caracterizadas por una propiedad cuya definición es difícilmente verificable, de modo que no existen en este momento ejemplos explícitos de variedades de contacto flexibles ni criterios geométricos caracterizándolas. El Capítulo 3 soluciona esta situación proponiendo un criterio geométrico en función de tres piezas fundamentales de la topología de contacto: los entornos de subvariedades, los nudos Legendrianos flexibles y las descomposiciones en libro abierto. En detalle, se prueba que una variedad de contacto es flexible si y sólo si existen subvariedades flexibles con un entorno suficientemente grande, que a su vez equivale a la existencia de una desestabilización del nudo

Legendriano trivial, lo cual se prueba que es una condición necesaria y suficiente para admitir una descomposición en libro abierto estabilizada negativamente.

El criterio geométrico probado al inicio del Capítulo 3 tiene una generosa lista de aplicaciones. Desarrollamos ciertas instancias a lo largo del capítulo, en particular centrándose en resultados sobre el grupo de contactomorfismos y la noción de orderabilidad introducida por Y. Eliashberg y L. Polterovich. Este Capítulo contiene a su vez resultados en topología de contacto posteriores a otros también obtenidos como parte de esta disertación. En particular ciertas ideas presentadas en el Capítulo 3 aparecen con sus debidas variantes en los capítulos siguientes.

La tercera parte de la disertación se desarrolla en los Capítulos 4, 5, 6, 7 y 8 y consiste en una serie de resultados estableciendo construcciones en topología de contacto que resuelven problemas antes abiertos y relacionando nociones en topología de contacto. Los Capítulos 4 y 7 sirven como instancias de construcciones introducidas en el campo de la topología de contacto que a su vez permiten la resolución de ciertos problemas en el ámbito. Los Capítulos 5, 6 y 8 relacionan respectivamente la noción de variedad de contacto flexible con la propiedad de orderabilidad, cotas contínuas para las funciones generatrices de contactomorfismos y estructuras simplécticas exóticas.

El Capítulo 1 sirve de introducción a la disertación, resaltando los aspectos fundamentales expuestos en cada capítulo e indicando al lector las ideas que subyacen los resultados presentados. Del mismo modo, incluye una explicación de la cronología del desarrollo científico e información básica sobre la organización de los capítulos.

En conclusión, este trabajo presenta aportaciones en el campo de la topología simpléctica y de contacto que han sido reconocidas por la comunidad matemática internacional. Parte de los capítulos han sido publicados en revistas de investigación y la disertación en su completitud ha sido presentada en congresos y seminarios en varias localizaciones y delante de expertos internacionales. En este momento, agradezco a la Universidad Autónoma de Madrid y al Instituto de Ciencias Matemáticas del Consejo Superior de Investigaciones Científicas la oportunidad de realizar este trabajo de investigación de matemáticas, esperando que su exposición invite al lector a desarrollar las ideas contenidas en esta disertación.

Se expone a continuación un listado de los trabajos de investigación desarrollados por el autor de esta disertación junto a sus coautores durante su periodo de formación investigadora:

1. Almost contact 5–manifolds are contact.
   junto a D. Pancholi y F. Presas,
   Aceptado en **Annals of Mathematics**.

2. A remark on the Reeb flow for spheres, junto a F. Presas,
   Aceptado en **Journal of Symplectic Geometry**.

3. On the non-existence of small positive loops
   of contactomorphisms on overtwisted contact manifolds.
   junto a F. Presas y S. Sandon,
   Aceptado en **Journal of Symplectic Geometry**.

4. h–principle for 4–dimensional contact foliations.
   junto a Á. Del Pino y F. Presas,
   Aceptado en **Int. Math. Res. Notices**.

5. Contact blow–up, junto a D. Pancholi y F. Presas,
   Aceptado en **Expositiones Mathematicae**.

6. Overtwisted Disks and Exotic Symplectic Structures.
   Enviado a **Bull. Société Mathématique de France**.

7. On the strong orderability of overtwisted 3–folds.
   junto a F. Presas, enviado a **Commentarii Helvetici**.

8. Chern-Weil theory and the group of strict contactomorphisms.
   junto a O. Spáčil, enviado a **Journal of Top. and Analysis**.

9. Higher Dimensional Maslov Indices
   junto a V.L. Ginzburg y F. Presas.

10. Geometric criteria for overtwistedness
    junto a E. Murphy y F. Presas.

CHAPTER 1

Introduction

Let us start with an example: consider a smooth function $f(q)$ in one variable $q \in [0,1]$ with a positive slope $p = f'(q) > 0$. Then the inequality $f(1) > f(0)$ holds, this is called Rolle's theorem. Suppose we are interested in a real valued function and its first derivative, we gather this information in the variables $(q,p,z) = (q, f'(q), f(q)) \in [0,1]\times\mathbb{R}\times\mathbb{R}$. In particular the function defines the embedded curve

$$\gamma_f(q) = \{(q, f'(q), f(q)) : q \in [0,1]\} \subseteq \mathbb{R}^3.$$

For a generic embedded curve $\gamma(t) = \{(q(t), p(t), z(t))\} \subseteq \mathbb{R}^3$ there does not exist $f \in C^1(\mathbb{R})$ such that $\gamma = \gamma_f$. A first necessary condition is that γ satisfies $z'(t) - p(t)q'(t) = 0$, let us call such embedded curves $\gamma \subseteq \mathbb{R}^3$ Legendrian curves.

Then the guiding principle for real functions and their first derivatives reads: Legendrian curves are what functions should have been. Briefly, the study of holomorphic functions and their integrals shows its true potential when the proper domains, Riemann surfaces, are considered. Even earlier, affine geometry naturally completes to projective geometry and these geometries lead to the fact that the space of functions is, in this sense, inappropriate: projectively, the only relevant facts about a function have to do with vanishing. In projective terms, hypersurfaces are what functions should have been. In our case we study functions and their first derivatives and the restriction to curves of the form γ_f is quite meaningless from a geometric viewpoint.

That being said, the space of embedded curves in $\mathbb{R}^3$ is an excessive definition for a function. Legendrian curves are the right ensemble, and a first compelling instance of this is the following fact. Given a family of Legendrian curves $\{\gamma_t\} \subseteq \mathbb{R}^3$, if $\gamma_0 = \gamma_f$ then any γ_t satisfies Rolle's theorem. This was proven by Y. Eliashberg [**44**], and the study of Legendrian curves conforms part of the origins of contact topology.

In particular, Rolle's theorem holds for a certain class of Legendrians. It can be said that the rigidity displayed by graphical Legendrians γ_f

persists after a Legendrian deformation. Nevertheless, Rolle's theorem for a general Legendrian is sheer nonsense. Indeed Figure 1 dismantles any strategy to prove a similar statement.

FIGURE 1. Failure of Rolle's theorem.

Legendrian curves whose graphs $\{(q, z(q))\} \subseteq \mathbb{R}^2$ contain such zigzag are tightly related to a class of flexible objects which appear in Chapter 3, the loose Legendrian submanifolds.

Still, Legendrian curves are defective due to the choice of coordinates. Thus it is only natural to study Legendrian curves up to the action of the coordinate transformations preserving the condition $\{p(q) - z'(q) = 0\}$. These transformations are called contactomorphisms and, as does any set of symmetries, they form a group. This transformation group has a prominent role in contact topology [**72**] and features in Chapters 5, 6 and 7.

The previous discussion leads to the definition of a contact structure. A contact structure on a smooth manifold is a hyperplane tangent distribution which can locally be expressed as $\{dz - pdq = 0\}$. In the higher dimensional case, the form pdq is understood to be a Liouville form on the symplectic space $(\mathbb{R}^{2n}, \omega_0)$.

It is enlightening to read the articles during the early stages of contact and symplectic topology, for instance those written by V.I. Arnol'd. The conjectures that have attracted activity in the field are often generalized statements of observations about functions, stemming either from Sturm–Liouville or Morse theory. Comprehensive accounts of this are [**3, 4, 5, 8**], or the shorter texts [**6, 7**]. In short, the theory of generating functions, Weinstein's Lagrangian creed and the Arnol'd conjectures are basic statements in the calculus of functions, where Legendrians (and Lagrangians) are what functions should have been.

Both symplectic and contact topology evolve from many areas of mathematics. For instance, Gibbs' [**69**] geometric treatise on thermodynamics can serve as beautiful motivation for the study of contact topology. The list is extensive, as can be grasped by the following words of V.I. Arnol'd on symplectic, and equivalently contact, topology:

"Symplectic geometry is the product of a long evolution of such branches of mathematics, as the variational calculus, the theory of dynamical systems, especially of Hamilton systems of classical mechanics, geometrical

optics, the theory of wave propagation, the study of the short waves or quasiclassical asymptotics in quantum mechanics, microlocal analysis of PDEs and the Lie theory of diffeomorphism groups and Poisson algebras".

From a modern perspective, the foundational results developed in symplectic and contact topology also provided a transition from the bow–and–arrows period to the tanks era [**9**]. This time the creative work of Y. Eliashberg, A. Floer, M. Gromov and many others yielded a powerful machinery, including a wealth of homology theories defined by pseudo-holomorphic curves, in favour of rigid results and also a meaningful development of the h–principle and the flexible features of these topologies. Instead of an extensive list of research articles, we refer the reader to the instructive accounts [**36, 51**].

1. Results of This Dissertation

Let $(Y, \ker \alpha_0)$ be a contact manifold, this dissertation is the result of studying the properties of the contact structures on the product smooth manifold $Y \times D^2(\rho)$. Suppose that $(Y \times D^2(\rho), \xi)$ is a contact structure that restricts to $(Y, \ker \alpha_0)$ on the submanifolds of the form $Y \times \{pt\}$, then a radial trivialization with the appropriate connection expresses the contact structure ξ as the kernel of a 1–form $\alpha \in \Omega^1(Y \times D^2(\rho))$ of the form

$$\alpha = \alpha_0 + H(p, r, \theta)d\theta, \text{ with } H : Y \times D^2(\rho) \longrightarrow \mathbb{R} \quad \text{s.t. } \partial_r H > 0.$$

This construction is first explained in Chapter 2 and further detailed in Chapter 7. The different Chapters in this dissertation are essentially based on questions regarding contact structures $(Y \times D^2, \xi)$ given by 1–forms of the form $\alpha_0 + H(p, r, \theta)d\theta$. Briefly, we now relate the core idea in each Chapter with the contact topology in $Y \times D^2(\rho)$.

1.1. Chapter 2. Suppose that we have a germ of a contact structure $(Y \times \partial D^2(\rho), \xi)$ such that the submanifolds $Y \times \{pt\}$ are contactomorphic to $(Y, \ker \alpha_0)$. The question motivating Chapter 2 reads:

Does $(Y \times \partial D^2(\rho), \xi)$ extend to a contact structure $(Y \times D^2(\rho), \xi)$?

It is quite clear that this is the case if $H(p, \rho, \theta) > 0$, and a positive answer to this question for general functions H would potentially yield a proof for the existence of contact structures in almost contact manifolds.

In Chapter 2 we use the space of contact elements of a 3–fold $(Y, \ker \alpha_0)$ in order to construct a contact structure $(Y \times \partial D^2(\rho), \xi)$ restricting to any given germ on the boundary. Then the existence problem for 5–folds is reduced to this case using [**45**] and Donaldson–Ibort–Martínez–Presas Lefschetz pencils.

The strategy followed in [**15**] is studying the question above in the particular case $(Y, \ker\alpha_0) \cong (\mathbb{R}^{2n-1}, \xi_0)$. This is indeed a better idea since the contact topology of $(\mathbb{R}^{2n-1}, \xi_0)$ is much more explicit and allows for a generous wealth of arguments that do not work for a general $(Y, \ker\alpha_0)$.

1.2. Chapter 3. The articles [**15, 45**] established a dichomoty between tight and overtwisted contact manifolds. The question motivating Chapter 3 can be stated as follows:

$$\text{Is } (Y \times D^2(\rho), \ker(\alpha_0 + r^2 d\theta)) \text{ tight or overtwisted ?}$$

The answer could certainly depend on whether $(Y, \ker\alpha_0)$ is itself tight or overtwisted, and it does. In fact, it also depends on the radius $\rho \in \mathbb{R}^+$, which is to be expected [**48**]. Chapter 3 provides answers to the above question and explores possible applications.

1.3. Chapter 4. Suppose that $(Y, \ker\alpha_0)$ is a contact sphere and we consider the contact manifold $(Y \times \dot{D}^2(\rho), \ker(\alpha_0 + r^2 d\theta))$. Since the disk is punctured we can write $(\mathbb{S}^{2n-1} \times \mathbb{S}^1_\rho \times \mathbb{R}^+, \ker(\alpha_0 + r^2 d\theta))$ and the basic observation in surgery theory is that the factor $\mathbb{R}^+$ can both be considered as a radius in $\mathbb{S}^{2n-1}$ or $\mathbb{S}^1$.

This is the main idea underlying Chapter 4: for instance, a small neighborhood of a codimension–2 submanifold with trivial normal bundle $(Y, \ker\alpha_0) \subseteq (M, \xi)$ can be written as $(Y \times D^2(\rho), \ker(\alpha_0 + r^2 d\theta))$ for $\rho \in \mathbb{R}^+$ small enough. Then the contact blow–up does erase the core $Y \times \{0\}$ and compactifies the punctured submanifold

$$(Y \times \dot{D}^2(\rho), \ker(\alpha_0 + r^2 d\theta))$$

using the radial direction of $\dot{D}^2(\rho)$ and the contact topology of $(Y, \ker\alpha_0)$. This is the seed for the constructions explained in Chapter 4.

1.4. Chapter 5. In order to obtain contact structures $(Y \times D^2(\rho), \xi)$, it is reasonable to study contact structures $(Y \times \mathbb{S}^2, \xi)$ and then remove one submanifold $Y \times \{pt\}$. The inverse procedure is precisely the question motivating a significant part of the Chapters 5, 6 and 7 in this dissertation:

$$\text{Can } (Y \times D^2, \xi) \text{ be compactified to } (Y \times \mathbb{S}^2, \xi) \text{ ?}$$

It is first reasonable to restrict to the hypothesis that the submanifolds $Y \times \{pt\}$ are contact submanifolds. Then a striking fact takes place: there exist contact manifolds $(Y, \ker \alpha_0)$ such that no contact structure $(Y \times \mathbb{S}^2, \xi)$ restricts to $(Y, \ker \alpha_0)$ on each $Y \times \{pt\}$. Nevertheless, the contact structure $(Y \times \mathbb{D}^2, \ker(\alpha_0 + r^2 d\theta))$ is available for the disk.

The existence of a contact structure on $(Y \times \mathbb{S}^2, \xi)$ with contact submanifolds $Y \times \{pt\}$ is intimately related to the (non)orderability of the contactomorphism group $\mathrm{Cont}(Y, \ker \alpha_0)$ [**72**].

There are examples of tight contact manifolds $(Y, \ker \alpha_0)$ and contact structures $(Y \times \mathbb{S}^2, \xi)$ with $(Y \times \{pt\}, \xi)$ contactomorphic to $(Y, \ker \alpha_0)$. There are also examples of tight contact manifolds where such a contact structure ξ cannot exist.

The question motivating Chapter 5 reads:

> Does there exists a contact structure $(Y \times \mathbb{S}^2, \xi)$ such that the submanifolds $(Y \times \{pt\}, \xi|_{Y \times \{pt\}})$ are overtwisted ?

Chapter 5 addresses this question providing a partial answer for certain overtwisted contact 3–folds (Y, ξ_0).

1.5. Chapter 6. Let $(Y, \ker \alpha_0)$ be a contact manifold and consider

$$(Y \times D^2(\rho), \ker(\alpha_0 + H(p, r, \theta) d\theta).$$

This is a contact manifold if and only if $\partial_r H > 0$. In particular any positive function $H \in C^\infty(Y \times \mathbb{S}^1)$, defines the contact structure

$$(Y_H, \xi_H) = (Y \times D^2(\rho), \ker(\alpha_0 + H(p, \theta) r^2 d\theta)),$$

for a certain $\rho \in \mathbb{R}^+$. Chapter 6 stems from the study of the contact structures (Y_H, ξ_H) and the certainly deep observation that a function $H \in C^\infty(Y \times \mathbb{S}^1)$ can be considered as a 1–parametric family of functions $\{H_\theta\} \subseteq C^\infty(Y)$.

Functions $H \in C^\infty(Y)$ generate (vector fields which generate) symmetries of Y, and thus $\{H_\theta\}$ gives rise to a 1–parametric subgroup $\{\varphi_\theta\} \subseteq \mathrm{Cont}(Y, \ker \alpha_0)$. Then the core of Chapter 6 is the following fact: in case $\varphi_1 = id$, the contact manifold (Y_H, ξ_H) is (PS–)overtwisted.

The overtwistedness of (Y_H, ξ_H) implies that it cannot be embedded into a symplectically fillable manifold, and this places restrictions on the radius $\rho \in \mathbb{R}^+$.

1.6. Chapter 7. Consider a contact structure $(Y \times \mathbb{S}^2, \xi)$ such that the submanifolds $Y \times \{pt\}$ are contactomorphic to $(Y, \ker \alpha_0)$. In particular this produces a map $s : \mathbb{S}^2 \longrightarrow \mathcal{C}(Y, \ker \alpha_0)$ from a 2–sphere to the space of contact structures $\mathcal{C}(Y, \ker \alpha_0)$ isotopic to $\ker \alpha_0$.

Observe that $\mathcal{C}(Y, \ker \alpha_0)$ is acted by $\mathrm{Diff}(Y)$ and the isotropy group is isomorphic to $\mathrm{Cont}(Y, \ker \alpha_0)$. It is simple to prove that this defines a Serre fibration (and even a locally trivial fibration). In particular we can obtain information on the homotopy type of $\mathrm{Cont}(Y, \ker \alpha_0)$ if we understand $\mathcal{C}(Y, \ker \alpha_0)$ and $\mathrm{Diff}(Y)$.

The central observation in Chapter 7 is that certain contact manifolds $(Y, \ker \alpha_0)$ can be endowed with a notable 2–sphere of contact structures. This idea follows the insights in [**80**] regarding twistor spaces: a hyperkähler structure on the symplectization provides a 2–sphere worth of contact structures on each level.

Then the connecting morphism $\partial : \pi_2 \mathcal{C}(Y, \ker \alpha_0) \longrightarrow \pi_1 \mathrm{Cont}(Y, \ker \alpha_0)$ is geometrically described using a contact connection and it is proven that the image $\partial(s)$ of these hyperkähler spheres $s \in \pi_2 \mathcal{C}(Y, \ker \alpha_0)$ are non–trivial elements of infinite order. The loop representing the class $\partial(s) \in \pi_1 \mathrm{Cont}(Y, \ker \alpha_0)$ is again related to the normal form

$$(Y \times D^2(\rho), \ker(\alpha_0 + H(p, r, \theta) d\theta))$$

presented above. In this case the loop of contactomorphisms is generated by the Hamiltonian $H(p, \rho, \theta)$ at the limit radius $r = \rho$.

1.7. Chapter 8. This chapter provides a simple answer to a simple question:

Is the symplectization of an overtwisted contact $(\mathbb{R}^{2n+1}, \xi)$ an exotic symplectic structure on $\mathbb{R}^{2n+2}$?

The answer is affirmative, and the argument is based on the following observation: since $(\mathbb{R}^{2n+1}, \ker \alpha)$ is overtwisted, the contact manifold $(\mathbb{R}^{2n+1} \times T^*\mathbb{S}^1, \ker(\alpha_0 + pdq))$ is also overtwisted. This manifold is the contactization of the symplectization of $(\mathbb{R}^{2n+1}, \ker \alpha)$, thus if the symplectization of $(\mathbb{R}^{2n+1}, \ker \alpha)$ embedded in standard $(\mathbb{R}^{2n+2}, \omega_0)$, its contactization would embed in standard $(\mathbb{R}^{2n+3}, \xi_0)$. In particular, it would be tight and contradict overtwistedness. Chapter 8 details this argument and explores these ideas.

This dissertation is a selection of the work developed during my graduate studies. There are three additional research articles not included in this thesis in order to preserve scientific coherence. These are [**25**], stemming from discussions with V.L. Ginzburg and F. Presas, [**26**], with Á. del Pino and F. Presas and the article [**27**], the results of which are part of O. Spáčil's thesis. The reader can hopefully enjoy reading them, in particular [**25**] and [**27**] are related to Chapter 7.

2. History of This Dissertation

The course of this dissertation does not follow its scientific order. Let me briefly explain the historical development of the presented results, which hopefully contributes to a better understanding for the reader. The essential ideas for the argument in Chapter 2 existed in April 2012. The strategy for proving existence of contact structures in every dimension required three steps:

a. Define a blow–up for (almost) contact Lefschetz pencils.
b. Find a contact structure on $F \times \mathbb{S}^2$ with certain properties.
c. Prove a 2–parametric version of the 5–dimensional result.

The first step lead to Chapter 4 and the second to Chapters 5, 6 and 7. In detail, the study of contact manifolds which contact fiber over the 2–sphere is in correspondence with (contractible) positive loops of contactomorphims. Thus working on the existence problem raised my interest for such loops.

The geometric way of obtaining a loop from a contact fibration is the central ingredient in Chapter 7: it is a nice exercise in order to understand the locally trivial fibration that relates the group of contactomorphisms with the space of contact structures. Confer Section 1 for a short explanation. Then there exists an almost contact connection providing a parallel transport that allows us to compare fibers, and Gray's stability provides the connecting morphism in the homotopy exact sequence of the aforementioned fibration.

The contact manifold (F, ξ) referred to in Step b. above is overtwisted in the argument given in Chapter 2. The mere existence of a positive loop in an overtwisted manifold is an open problem. Many experts in the field conjectured that no such loop could exist, and in fact F. Presas had a strategy to prove this. Although the argument he proposed would contradict a version of one of Arnol'd's conjecture, a part of the strategy implied a continuous lower bound for the generating Hamiltonian. This idea is the seed for Chapter 6, where it is developed in combination with contact ambient surgeries to obtain the lower bound for a general overtwisted 3–fold.

It follows a brief chronology of the events conforming this dissertation:

04/2012	Chapter 2. Existence for 5–folds
10/2012	Chapter 4. Contact blow–up
02/2013	Chapter 7. Loops at infinity
10/2013	Chapter 6. C^0–lower bounds
03/2014	Chapter 8. Exotic symplectizations
06/2014	Chapter 5. Positive loops and Lutz twists
12/2014	Chapter 3. Geometric criteria for Overtwistedness

During 2013, I also became convinced that overtwisted manifolds could have positive loops of contactomorphims. The reason for this were the (quaternionic) symmetries presented by the unique tight contact structure $(\mathbb{S}^1 \times \mathbb{S}^2, \xi)$. I had used these symmetries in the results of Chapter 7 and the contact manifold $(\mathbb{S}^1 \times \mathbb{S}^2, \xi)$, albeit tight, was to me quite close to an overtwisted closed manifold. It thus started the time to prove the opposite of the statement we had long been trying to prove. Then, discussions with F. Presas lead to the arguments presented in Chapter 5, where positive loops of contactomorphims are build in certain overtwisted 3–folds.

Regarding the content of Chapter 8, it stems from conversations with S. Courte and E. Giroux at the École Normale de Lyon. E. Giroux and M. Mazzucchelli had kindly invited me to talk in the geometry seminar, and during a tea break we thought about whether the symplectization of an overtwisted contact $\mathbb{R}^3$ was an exotic symplectic $\mathbb{R}^4$. This question was related to part of S. Courte's beautiful thesis [**40**], and after a discussion with him the arguments presented in Chapter 8 proved that it was indeed the case.

The proof of the existence of a contact structure in every almost contact manifold was presented during April 2014 in [**15**], and Chapter 3 is written after the first appearance of this article. The following section addresses this issue.

3. History of This Dissertation II

This Dissertation contains results in the area of contact and symplectic topology. This is a rapidly growing field with many creative experts constantly expanding its boundaries. In consequence, a part of the results in this dissertation have been either reproven or generalized.

The central instance of this is the foundational article [**15**], which vastly generalizes the results of Chapter 2. It is certainly exciting to be a witness of such essential progress and Chapter 3 grows from many valuable discussions with its authors M.S. Borman, Y. Eliashberg and E. Murphy.

In the same vein, E. Giroux had a simpler proof for one of the results in Chapter 7 which had been again reproven after a discussion with V.L. Ginzburg [**25**] and with O. Spacil [**27**].

The chronology of the work developed in this dissertation is relevant for the reader and partially justifies its exposition. In December 2014 I had already realized that the Chapters 6 and 8 are largely implied by Chapter 3. Hence it would seem unfair to force the reader through the Chapters 6 and 8 when it suffices to understand Chapter 3. Thus Chapter 3 is located earlier in the dissertation.

In addition, Chapters 4 to 8 are based in at most two ideas each, thus making them more suitable for a later reading. In contrast, Chapters 2 and 3 might require proper concentration, and scientifically constitute the hard core of this work. Hence their location in this dissertation.

4. Organization

This dissertation gathers ideas and results during my graduate years in Madrid, which comprise roughly three years. Section 1 links the different results in the common framework provided by the study of contact structures $(Y \times D^2, \xi)$. However, the reader is probably interested in specific chapters of this dissertation. Thus it seems adequate to provide self–contained accounts for the results presented in each chapter. In consequence, the chapters contain their respective references and each includes an introduction detailing the context in which the results can be better understood.

In short, the reader should be able to read a chapter with no previous knowledge on the others. It does hopefully help to read the previous Section 1, but the essential context and preliminaries are covered in each chapter.

5. Background

The chapters in this dissertation are largely self–contained. Except for Chapters 2 and 3, the only requirement is a mild knowledge of contact topology as gathered in [**4, 8, 61**]. This should suffice for a mathematical understanding of the results contained therein. Regarding the relevance of each particular result in the context of contact and symplectic topology, the appropriate references are provided in each chapter.

Chapter 2 is partially self–contained. It does use significant results coming from [**42, 43, 45**], however the definitions and statements required for the proof of the main result are reviewed in the beginning of the chapter. Thus the reader is hopefully able to understand the statements of these results, even if he has not entirely comprehended the foundations preceding them.

Chapter 3 is presumably the most demanding for the reader. In fairness, it also contains the most meaningful results. On the one hand, it would be possible to write a comprehensive account with the required background, but it would considerably lengthen this dissertation. And on the other, there are excellent references in the literature to which we can refer the reader. In particular, a good understanding of the works [**15, 71**] and the book [**36**, Part 2] should suffice.

6. Acknowledgements

I would like to thank my academic supervisor, Francisco Presas, for introducing me to contact and symplectic topology and his guidance and support during my graduate years. He has been generous with his time and largely open to discussions in any topics, from which I have learned many interesting mathematics.

I am extremely grateful to the many mathematicians working on contact and symplectic topology. This community provides a stimulating environment through the regularly organized conferences and schools with a remarkable scientific quality (and delightfully chosen locations).

In particular, I would like to acknowledge the several researchers that have invited me to participate in their seminars. I would especially like to thank the following researchers from which I have learnt a significant amount of mathematics, these are Y. Eliashberg, V.L. Ginzburg, E. Giroux, P. Massot, K. Niederkrüger and O. van Koert.

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CHAPTER 2

Almost contact 5–manifolds are contact

In this second chapter we prove the existence of a contact structure in any homotopy class of almost contact structures on a closed 5–dimensional manifold. This result is joint work with D.M. Pancholi and F. Presas.

1. Introduction

Let (M^{2n+1}, ξ) be a cooriented contact manifold with associated contact form α, i.e. $\xi = \ker \alpha$ such that $\alpha \wedge d\alpha^n \neq 0$. This structure determines a symplectic distribution $(\xi, d\alpha_{|\xi}) \subset TM$. Any change of the associated contact form α does not change the conformal symplectic class of $d\alpha$ restricted to ξ. This allows us to choose a compatible almost complex structure $J \in End(\xi)$. Thus given a cooriented contact structure we obtain in a natural way a reduction of the structure group $Gl(2n+1, \mathbb{R})$ of the tangent bundle TM to the group $U(n) \times \{1\}$, which is unique up to homotopy, see [**61**, Prop. 2.4.8]. A manifold M is said to be an *almost contact manifold* if the structure group of its tangent bundle can be reduced to $U(n) \times \{1\}$. In particular, cooriented contact manifolds are almost contact manifolds and such a reduction of the structure group of the tangent bundle of a manifold M is a necessary condition for the existence of a cooriented contact structure on M. In 2012, it was unknown whether this condition was in general sufficient. The reader should however read the recent development [**15**], which also features prominently in Chapter 3 in this dissertation, and observe that the proof presented in this Chapter precedes in two years the article [**15**].

There are classical cases in which the existence of an almost contact structure is sufficient for the manifold to admit a contact structure. For example, if the manifold M is open then one can apply Gromov's h–principle techniques to conclude that the condition is sufficient. See the result 10.3.2 in [**54**]. The scenario is quite different for closed almost contact manifolds. Using results of Lutz [**92**] and Martinet [**94**] one can show that every cooriented tangent 2–plane field on a closed oriented 3–manifold is homotopic to a contact structure. A good account of this result from a modern perspective is given in [**61**]. For manifolds of higher dimensions there are various results establishing the sufficiency of the condition. Important instances of these are the construction of contact structures on certain principal $\mathbb{S}^1$–bundles over closed symplectic manifolds due to Boothby and Wang [**17**], the existence of a contact structure on the product of a contact manifold with a surface of genus greater than zero following Bourgeois [**19**] and the existence of contact structures on simply connected 5–dimensional closed orientable manifolds obtained by Geiges [**62**] and its higher dimensional analogue [**63**].

Let us turn our attention to 5–manifolds since the main goal of this Chapter is to show that any orientable almost contact 5–manifold is contact. In this case H. Geiges has been studying existence results in other situations apart from the simply connected one. In [**66**] a positive result is also given for spin closed manifolds with $\pi_1 = \mathbb{Z}_2$, and spin closed manifolds with finite fundamental group of odd order are studied in [**67**]. On the other hand there is also a construction of contact structures on an orientable 5–manifold occurring as a product of two lower dimensional manifolds by Geiges and Stipsicz [**68**]. While Geiges used the topological classification of simply connected manifolds for his results in [**62**], one of the ingredients in [**68**] is a decomposition result of a 4–manifold into two Stein manifolds with common contact boundary [**1, 12**].

Being an almost contact manifold is a purely topological condition. In fact, the reduction of the structure group can be studied via obstruction theory. For example, in the 5–dimensional situation a manifold M is almost contact if and only if the third integral Steifel–Whitney class $W_3(M)$ vanishes. Actually, using this hypothesis and the classification of simply connected manifolds due to D. Barden [**10**], H. Geiges deduces that any manifold with $W_3(M) = 0$ can be obtained by Legendrian surgery from certain model contact manifolds. Though this approach is elegant, it seems quite difficult to extend these ideas to produce contact structures on any almost contact 5–manifold. We therefore propose a different approach: the existence of an *almost contact pencil* structure on the given almost contact manifold is the required topological property to produce a contact structure. The tools appearing in our proof use techniques from three different sources:

- The approximately holomorphic techniques developed by Donaldson in the symplectic setting [**42, 43**] and adapted in [**83, 114**] to the contact setting to produce the so–called *quasi contact pencil*.

- Eliashberg's classification of overtwisted 3–dimensional manifolds [**45**] to produce overtwisted contact structures on the fibres of the pencil.

- The canonical structure of the space of contact elements in a 3–manifold. See [**93**].

Let us state the main result in this Chapter.

THEOREM 1.1. *Let M be a closed oriented 5–dimensional manifold. There exists a contact structure in every homotopy class of almost contact structures.*

In particular closed oriented almost contact 5–manifolds are contact. It is important to emphasize that using the techniques developed in this Chapter, it is not possible to conclude anything about the number of distinct contact distributions that may occur in a given homotopy class of almost contact distributions. The result states that there is at least one, the article [**113**] provides examples with more. It follows from the construction that the contact structure is PS–overtwisted [**106, 107**] and therefore it is non–fillable.

REMARK 1.1. The data given by an almost contact structure is tantamount to that of a hyperplane subbundle of the tangent bundle endowed with a complex structure [**61**]. An almost contact structure will refer to either the reduction of the structure group or to such distribution. In the course of the Chapter the distributions are supposed to be coorientable and Section 10 contains the corresponding results for non–coorientable distributions.

The proof of Theorem 1.1 consists of a constructive argument in which we obtain the contact condition step by step. These steps correspond to the sections of the paper as follows:

- To begin with, we explain how to produce over any almost contact 5–manifold (M, ξ) an almost contact fibration over $\mathbb{S}^2$ with singularities of some standard type. It is defined on the complement of a link. The definition and properties of this almost contact fibration – in fact, an almost contact pencil – is the content of Sections 2 and 3. The details of the actual construction are not provided and the reader is referred to [**82, 95, 114**] for the proofs. The existence of such a pencil is the input data of this Chapter.

- In Section 4, we produce a first deformation of the almost contact structure ξ to obtain a contact structure in a neighborhood of the singularities of the fibration and in a neighborhood of the link.

- The neighborhood of the link has the structure of a base locus of a pencil occurring in algebraic or symplectic geometry. In order to provide a Lefschetz type fibration we blow–up the base locus. This requires the notion of a contact blow–up. For the purposes of this Chapter, it will be enough to define an appropriate contact surgery of the 5–manifold along a transverse $\mathbb{S}^1$. This is the content of Section 5.

- Away from the critical points the distribution splits as $\xi = \xi_v \oplus \mathcal{H}$, where ξ_v is the restriction of the distribution to the fibres and $\mathcal{H}$ is the symplectic orthogonal. Section 6 deals with a deformation of ξ_v to produce a contact structure in the fibres. It strongly uses the classification of overtwisted contact manifolds due to Eliashberg [**45**].

- In Section 7 we begin to deform the horizontal direction $\mathcal{H}$. This is done in two steps. Given a suitable cell decomposition of the base $\mathbb{S}^2$, we first deform $\mathcal{H}$ in the pre–image of a neighborhood of the 1–skeleton. Section 7 contains this first step.

- The contact condition still has to be achieved in the pre–image of the 2–cells. This is the second step. The contact structure used in order to fill the pre–image of the 2–cells is constructed in Section 8. This construction uses the contact structure of the space of contact elements of the 3–dimensional fibre.

- In Section 9 we obtain a contact structure on the surgered 5–manifold using the results obtained in Section 8. Then we reverse the blow–up surgery and construct the contact structure on the initial 5–manifold. Theorem 1.1 is concluded.

- In Section 10 we deal with the case of non–coorientable distributions. We introduce the suitable definitions and explain the non–coorientable version of Theorem 1.1.

The more technical results on this Chapter are contained on Sections 5, 6 and 8. Section 7 (resp. Section 9) is also essential but the exposition can be made less technical and the reader should be able to readily comprehend it once Sections 5 and 6 (resp. Section 8) are understood. Section 6 and 7 can be understood without Section 5 and Section 8 can be read almost independently.

The work in this Chapter was presented in the Spring 2012 AIM Workshop on higher dimensional contact geometry. In its course, J. Etnyre commented on a possible alternative approach in the framework of Giroux's program using an open book decomposition. The argument has been subsequently written and it is the content of the article [**57**].

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2. Preliminaries.

2.1. Quasi–contact structures. Let M be an almost contact manifold. There exists a choice of a symplectic distribution $(\xi, \omega) \subset TM$ for such a manifold. Namely, we can find a 2–form η on ξ with the property that η is non–degenerate and compatible with the almost complex structure J defined on ξ. By extending η to a form on M we can find a 2–form ω on M such that $(\xi, \omega_{|\xi})$ becomes a symplectic vector bundle. This form ω is not necessarily closed. The triple (M, ξ, ω) is also said to be an almost contact manifold. In other words, an almost contact structure is meant to be a triple (ξ, J, ω) for some ω as discussed. The choice of almost complex structure J is homotopically unique and it might be omitted. An almost contact manifold is subsequently described by a triple (M, ξ, ω).

In order to construct a contact structure out of an almost contact one, the first step is to provide a better 2–form on M. That is, we replace ω by a closed 2–form.

DEFINITION 2.1. A manifold M^{2n+1} admits a *quasi–contact structure* if there exists a pair (ξ, ω) such that ξ is a codimension 1–distribution and ω is a closed 2–form on M which is non–degenerate when restricted to ξ.

Notice that a quasi–contact pair (ξ, ω) admits a compatible almost contact structure, i.e. there exists a J which makes (ξ, J, ω) into an almost contact structure. These manifolds have also been called 2*–calibrated* [**81**] in the literature. The following lemma justifies the appearance of the previous definition:

LEMMA 2.2. *Every almost contact manifold (M, ξ_0, ω_0) admits a quasi–contact structure (ξ_1, ω_1) homotopic to (ξ_0, ω_0) through symplectic distributions and the class $[\omega_1]$ can be fixed to be any prescribed cohomology class $a \in H^2(M, \mathbb{R})$.*

PROOF. Let $j : M \longrightarrow M \times \mathbb{R}$ be the inclusion as the zero section. Consider a not–necessarily closed 2–form $\widetilde{\omega}_0$, such that $\omega_0 = j^*\widetilde{\omega}_0$. Fix a Riemannian metric g over M such that ξ_0 and $\ker \omega_0$ are g–orthogonal.

Apply Gromov's classification result of open symplectic manifolds to produce a 1–parametric family $\{\widetilde{\omega}_t\}_{t=0}^1$ of *symplectic* forms such that for $t = 1$ the form is closed. See [**54**], Corollary 10.2.2. Let $\pi : M \times \mathbb{R} \longrightarrow M$ be the projection and choose the cohomology class defined by $\widetilde{\omega}_1$ to be π^*a. Consider the family of 2–forms $\omega_t = j^*\widetilde{\omega}_t$ on M. Since $\widetilde{\omega}_t$ is non–degenerate on $M \times \mathbb{R}$ for each t, the form ω_t has 1–dimensional kernel $\ker \omega_t$. Define $\xi_t = (\ker \omega_t)^{\perp g}$. Then (ξ_t, ω_t) provides the required family. $\square$

This is the farthest one can reach by the standard h–principle argument in order to find contact structures on a closed manifold. One can start with the almost contact bundle $\xi = \ker \alpha$ and use Lemma 2.2 to find a 2–form $d\beta$ such that $(\xi, d\beta)$ is a symplectic bundle, but there is no general method to achieve α equal to β. This is the aim of the Chapter.

2.2. Obstruction theory. The content of Theorem 1.1 has two parts. The statement implies the existence of a contact structure in an almost contact manifold. This is a result in itself, regardless of the homotopy type of the resulting almost contact distribution. The construction we provide in this Chapter also concludes that the obtained contact distribution lies in the same homotopy class of almost contact distributions as the original almost contact structure. This is achieved via the study of an obstruction class. Let us review some well–known facts.

Let M be a smooth oriented 5–manifold and $\pi : TM \longrightarrow M$ its tangent bundle. The projection π is considered to be an $SO(5)$–principal frame bundle. An almost contact structure is a reduction of the structure group $G = SO(5)$ to a subgroup $H \cong U(2) \times \{1\} \cong U(2)$. The isomorphism classes of almost contact structures are parametrized by the homotopy classes of such reductions. A reduction of the structure group G to a subgroup H is tantamount to a section of a G/H–bundle over M. Hence the classification of almost contact structures on M is reduced to the study of homotopy classes of sections of a $SO(5)/U(2)$–bundle over M.

LEMMA 2.3. *There exists a diffeomorphism $SO(5)/U(2) \cong \mathbb{CP}^3$.*

See [**61**, Prop. 8.1.3] for the proof of this Lemma.

The homotopy groups $\pi_i(\mathbb{CP}^3) = 0$ for $1 \leq i \leq 6$, $i \neq 2$, hence the existence of sections of a fibre bundle with typical fibre $\mathbb{CP}^3$ over the 5–manifold M is controlled by the primary obstruction class $d = W_3(M) \in H^3(M, \pi_2(\mathbb{CP}^3)) \cong H^3(M, \mathbb{Z})$. The hypothesis of Theorem 1.1 is $d = 0$.

Let s_ξ and $s_{\xi'}$ be two sections of this $\mathbb{CP}^3$–bundle. The obstruction class dictating the existence (or the lack thereof) of a homotopy between them is the primary obstruction $d(\xi, \xi') \in H^2(M, \mathbb{Z})$. The obstruction theory argument can be made relative to a submanifold $A \subset M$. Given a self–indexing Morse function for the pair (M, A), we consider the relative j–skeleton M_j defined as the union of A and the cores of the handles of the critical points of index less or equal than j. We have the following

LEMMA 2.4. *Consider a relative 2–skeleton M_2 for the pair (M, A) and let s_ξ, $s_{\xi'}$ be two sections of a $\mathbb{CP}^3$–bundle over M that are homotopic over M_2. Then s_ξ and $s_{\xi'}$ are also homotopic over (M, A).*

Let (M, ξ) be an almost contact structure, the construction of the contact structure ξ' obtained in Theorem 1.1 does not modify the homotopy class of the given section, i.e. $s_\xi \sim s_{\xi'}$. In Section 8 we provide a detailed account on the modification of the obstruction class $d(\xi, \xi')$ in the 2–skeleton of certain pieces of M where ξ' has been constructed. This is enough to conclude that $d(\xi, \xi') = 0$ once ξ' is extended to M in Section 9.

2.3. Homotopy of vector bundles. The argument constructing the homotopy between the initial almost contact structure and the resulting contact distribution in Theorem 1.1 uses the following lemma. It is used in several parts of Sections 4 to 9.

Let (V, ω) be an oriented vector space of dimension $\dim_{\mathbb{R}} V = 4$. Consider an splitting $V = V_0 \oplus V_1$ with V_0, V_1 two oriented 2–dimensional vector subspaces. Since $Sp(2,\mathbb{R})/SO(2)$ is contractible, the space of symplectic structures on V such that V_0 and V_1 are symplectic orthogonal subspaces is contractible. This essentially implies the following

LEMMA 2.5. *Let M be an almost contact 5–manifold, A an open submanifold of M, and $(\xi_0, \omega_0), (\xi_1, \omega_1)$ two almost contact structures on M such that there exists a homotopy $\{\xi_t\}$ of oriented distributions on (M, A) connecting ξ_0 and ξ_1. Suppose that there exist L_0 and L_1 two rank–2 symplectic subbundles of ξ_0 and ξ_1 and a homotopy $\{L_t\} \subset \{\xi_t\}$ of oriented distributions connecting L_0 and L_1 on (M, A). Then there is a path $\{\omega_t\}$ of symplectic structures on $\{\xi_t\}$ such that $\{(\xi_t, \omega_t)\}$ is a path of almost contact structures connecting (ξ_0, ω_0) and (ξ_1, ω_1) on (M, A).*

PROOF. Consider J_0 and J_1 two compatible complex structures on the symplectic distributions ξ_0 and ξ_1 respectively. These define two fibrewise scalar–product structures

$$g_0 = \omega_0(\cdot, J_0\cdot) \text{ and } g_1 = \omega_1(\cdot, J_1\cdot)$$

on ξ_0 and ξ_1. The space of fibrewise scalar–product structures has contractible fibre, namely $Gl^+(4,\mathbb{R})/SO(4)$, and thus it is contractible. Hence, there exists a homotopy $\{g_t\}$ of fibrewise scalar–products connecting g_0 and g_1. The scalar–product g_t provides an orthogonal decomposition $\xi_t = L_t \oplus L_t^{\perp_{g_t}}$. The homotopy of oriented bundles $\{L_t\}$ induces a homotopy of oriented bundles $\{L_t^{\perp_{g_t}}\}$ respecting the symplectic splitting given by ω_0 and ω_1 on ξ_0 and ξ_1. $\square$

2.4. Notation. Let $\mathbb{R}^{2n}$ be Euclidean space, $B^{2n}(r) = \{p \in \mathbb{R}^{2n} : \|p\| \leq r\}$ denotes the closed ball of radius r centered at the origin. The 2–dimensional balls are also referred to as disks and denoted by $\mathbb{D}^2(r)$. In case the radius is omitted B^{2n} and $\mathbb{D}^2$ denote the ball and disk of radius 1 respectively.

3. Quasi–contact pencils.

Approximately holomorphic techniques have been extremely useful in symplectic geometry. Their main application in contact geometry – due to E. Giroux – is to establish the existence of a compatible open book for a contact manifold in higher dimensions. See [**37, 71, 114**]. An open book decomposition is a way of trivializing a contact manifold by fibering it over $\mathbb{S}^1$. Such objects have also been studied in the almost contact case, see [**96**].

There exists a construction [**112**] in the contact case analogous to the Lefschetz pencil decomposition introduced by Donaldson over a symplectic manifold [**43**]. It is called a contact pencil and it allows us to express a contact manifold as a singular fibration over $\mathbb{S}^2$. It has been extended in [**82, 95, 114**] to the quasi–contact setting. Theorem 3.1 and Corollary 3.6 in this Section provide the existence of a quasi–contact pencil with suitable properties. Let us begin with the appropriate definitions.

Definition 3.1. An almost contact submanifold of an almost contact manifold (M,ξ,ω) is an embedded submanifold $j : S \longrightarrow M$ such that the induced pair $(j^*\xi, j^*\omega)$ is an almost contact structure on S.

A quasi–contact submanifold of a quasi–contact manifold is defined analogously. In particular this implies in both cases that the submanifold S is transverse to the distribution ξ.

A chart $\phi : (U,p) \longrightarrow V \subset (\mathbb{C}^n \times \mathbb{R}, 0)$ of an atlas of M is compatible with the almost contact structure (ξ,ω) at a point $p \in U \subset M$ if the push–forward at p of ξ_p by ϕ is $\mathbb{C}^n \times \{0\}$ and the 2–form $\phi_*\omega(p)$ is a positive $(1,1)$–form with respect to the canonical almost complex structure.

Definition 3.2. An almost contact pencil on a closed almost contact manifold (M^{2n+1},ξ,ω) is a triple (f,B,C) consisting of a codimension–4 almost contact submanifold B, called the base locus, a finite set C of smooth transverse curves and a map $f : M\backslash B \longrightarrow \mathbb{CP}^1$ conforming the following conditions:

1. The map f is a submersion on the complement of C and the fibres $f^{-1}(p)$, for any $p \in \mathbb{CP}^1$, are almost contact submanifolds at the regular points.
2. The set $f(C)$ is a finite union of locally smooth curves with transverse self–intersections.

(3) At a critical point $p \in C \subset M$ there exists a compatible chart ϕ_p such that

$$(f \circ \phi_p^{-1})(z_1, \ldots, z_n, s) = f(p) + z_1^2 + \ldots + z_n^2 + g(s)$$

where $g : (\mathbb{R}, 0) \longrightarrow (\mathbb{C}, 0)$ is an immersion at the origin.

(4) Each $b \in B$ has a compatible chart to $(\mathbb{C}^n \times \mathbb{R}, 0)$ under which B is locally cut out by $\{z_1 = z_2 = 0\}$ and f corresponds to the projectivization of the first two coordinates, i.e. locally

$$f(z_1, \ldots, z_n, t) = \frac{z_2}{z_1}.$$

Remark 3.3. Quasi–contact pencils for quasi–contact manifolds and contact pencils for contact manifolds are defined by replacing the expression almost contact by the suitable one in each case.

The generic fibres of f are open almost contact submanifolds and the closures of the fibres at the base locus are smooth. This is because the local model (4) in the Definition 3.2 is a parametrized elliptic singularity and the fibres come in complex lines $\{z_2 = const \cdot z_1\}$ joining at the origin. We refer to the compactified fibres so constructed as the fibres of the pencil. See Figure 1.

In dimension 5, each compactified smooth fibre is a smooth 3–manifold containing B as a link and any two different compactified fibres intersect transversely along B. Note that if we remove a tubular neighborhood of C in M the compactified fibre over a neighborhood of a point in $f(C)$ becomes a smooth manifold whose boundary is a (union of) 2–tori. This boundary components can be filled by solid tori at any regular fibre.

Notice that the set of critical values $\Delta = f(C)$ are no longer points, as in the symplectic case, but immersed curves. This is because of Condition (3) in the Definition 3.2. In particular, the usual isotopy argument between two fibres does not apply unless their images are in the same connected component of $\mathbb{CP}^1 \backslash \Delta$. This has been studied in the contact and quasi–contact cases. The set C is a positive link and therefore Δ is also oriented. There is a partial order in the complement of Δ: a connected component P_0 is less or equal than a connected component P_1 if P_0 and P_1 can be connected by an oriented path $\gamma \subset \mathbb{CP}^1$ intersecting Δ only with positive crossings. The proposition that follows has only been proved for the contact and quasi–contact cases. An analogous statement probably remains true in the almost contact setting. It is

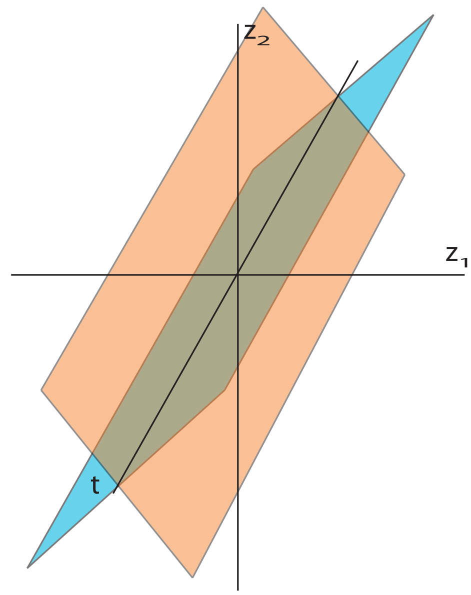


FIGURE 1. Fibres close to the base locus $B = \{z_1 = z_2 = 0\}$.

provided to offer some geometric insight about contact and quasi–contact pencils, it is not used in the rest of the Chapter.

PROPOSITION 3.4 (Proposition 6.1 of [**112**]). *Let M be a quasi–contact manifold equipped with a quasi–contact pencil (f, B, C). Then if two regular values of f, P_0 and P_1, are separated by a unique curve of Δ then the two corresponding fibres $F_0 = \overline{f^{-1}(P_0)}$ and $F_1 = \overline{f^{-1}(P_1)}$ are related by an index $n-1$ surgery.*

Suppose that the manifold and the pencil are contact, then the surgery is Legendrian and it attaches a Legendrian sphere to F_0 if P_0 is smaller than P_1. See Figure 2.

In the contact case it implies that the crossing of a singular curve in the fibration amounts to a directed Weinstein cobordism. In the quasi–contact case no such orientation appears. For instance, the case in which the quasi–contact distribution is a foliation – in dimension 3 this is a taut foliation – becomes absolutely symmetric and there is no difference in crossing one way or the other.

Examples. The following two constructions yield simple instances of contact pencils.

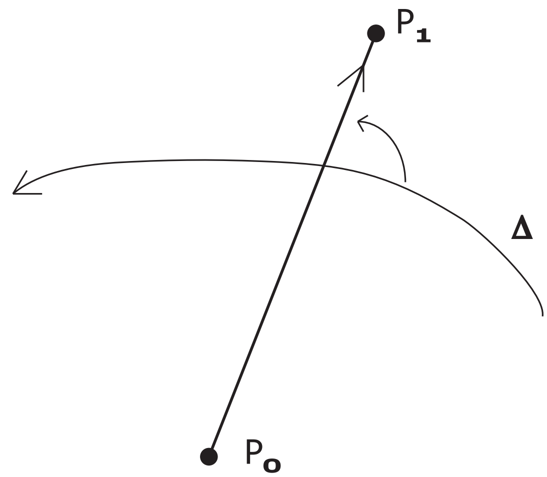


FIGURE 2. According to the orientations, the fibre $F_1 = \overline{f^{-1}(P_1)}$ is obtained via a Legendrian surgery on the fibre $F_0 = \overline{f^{-1}(P_0)}$.

1. Consider a closed symplectic manifold (M, ω) with $[\omega]$ of integral class and a symplectic Lefschetz pencil (f, B, C) on (M, ω) as constructed in [**43**]. Consider the circle bundle $\mathbb{S}(L)$ associated to ω with its Boothby–Wang contact structure $(\mathbb{S}(L), \xi_\omega)$, defined in [**17**], and the projection $\pi : \mathbb{S}(L) \longrightarrow M$. Then the triple
$$(\pi^* f, \pi^{-1}(B), \pi^{-1}(C))$$
is, after a small perturbation of $\pi^* f$, a contact pencil for $(\mathbb{S}(L), \xi_\omega)$.

2. Given two generic complex polynomials in $\mathbb{C}^n$ of high enough degree, we can construct the associated complex pencil (f, B, C). Suppose that the base points set B contains the origin and denote the standard embedding of the radius r sphere by $e_r : \mathbb{S}^{2n-1} \longrightarrow \mathbb{C}^n$. Then for a generic radius $\rho > 0$, the triple $(e_\rho^* f, e_\rho^{-1}(B), \mathrm{Crit}(e_\rho^*(f)))$ is a contact pencil for $(\mathbb{S}^{2n-1}, \xi_{st})$.

Consider a quasi–contact structure (M, ξ, ω). The main existence result [**82, 95, 114**] can be stated as

THEOREM 3.1. *Let (M, ξ, ω) be a quasi–contact manifold with $[\omega]$ rational. Given an integral cohomology class $a \in H^2(M, \mathbb{Z})$, there exists a* *quasi–contact pencil (f, B, C) such that the fibres are Poincaré dual to the class $a + k[\omega]$, for any $k \in \mathbb{N}$ large enough.*

The basic construction goes as follows. Consider a line bundle V whose first Chern class equals a and denote by L a Hermitian line bundle over M whose curvature is $-i\omega$. The pencil is constructed using a suitable approximately holomorphic section $\sigma_1^k \oplus \sigma_2^k : M \longrightarrow \mathbb{C}^2 \otimes (L^k \otimes V)$, this requires $k \in \mathbb{N}$ to be large enough. The pencil map is $f_k = [\sigma_1^k : \sigma_2^k] : M \backslash B_k \longrightarrow \mathbb{CP}^1$ and the base locus is $B_k = \{p \in M : \sigma_1^k(p) = \sigma_2^k(p) = 0\}$. A point $p \in M$ maps to $[\sigma_1^k(p) : \sigma_2^k(p)] \in \mathbb{CP}^1$. This is well–defined if p is not contained in the base locus B_k. The construction is detailed in [**114**].

The proof of this result does not work in the almost contact setting. In order to construct the pencil, the approximately holomorphic techniques are essential and for them to work we need the closedness of the 2–form ω (so as to be able to construct the line bundle L). In general, a quasi–contact pencil may have empty base locus. Nevertheless a pencil obtained through approximately holomorphic sections on a higher dimensional manifold does not.

The following lemma will be useful.

LEMMA 3.5. *Let (M, ξ, ω) be an almost contact 5–manifold, (f, B, C) an almost contact pencil adapted to it and obtained from a section $s_1 \oplus s_2$ of the bundle $\mathbb{C}^2 \otimes \det(\xi)$, and so the base locus is defined as $B = Z(s_1 \oplus s_2)$ and the pencil map is $f := [s_1 : s_2] : M \setminus B \to \mathbb{CP}^1$. Then the Chern class of ξ_F vanishes for any regular fibre (F, ξ_F).*

PROOF. Let F be a regular fibre of f, this fibre is defined as the zero set of the section $s_\lambda = \lambda_1 s_1 + \lambda_2 s_2$, for a fixed $[\lambda_1 : \lambda_2] \in \mathbb{CP}^1$. This is a section of the bundle $\det(\xi)$. Along this fibre F, the distribution ξ satisfies

$$c_1(\xi)|_F = c_1(\xi_F) + c_1(\nu_F).$$

The statement follows from $c_1(\nu_F) = c_1(\det \xi)|_F = c_1(\xi)|_F$ inserted in the previous equation. $\square$

In case the form ω of the quasi–contact structure is exact – then called an exact quasi-contact structure – we obtain the following

COROLLARY 3.6. *Let (M, ξ, ω) be an exact quasi–contact closed manifold. Then it admits a quasi–contact pencil such that any smooth fibre*

F satisfies $c_1(\xi_F) = 0$. Further, the base locus B is non–empty if $\dim M$ *is greater than* 3*.*

PROOF. We use Theorem 3.1 to construct a pencil such that the cohomology class $a \in H^2(M, \mathbb{Z})$ is fixed to be $a = c_1(\xi) = c_1(\det \xi)$. Since ω is exact, thus $L \cong \mathbb{C}$, we obtain that the section defining the pencil $s_1 \oplus s_2$ is a section of the bundle $\mathbb{C}^2 \otimes (\det \xi \otimes L^k) = \mathbb{C}^2 \otimes \det(\xi)$. Lemma 3.5 implies that the almost contact structure induced in the regular fibres of the pencil has vanishing first Chern class.

Let us prove the non–emptiness of the set B. It is explained in [**82, 83**] that the submanifold $B = Z(\sigma_1^k \oplus \sigma_2^k)$ satisfies a Lefschetz hyperplane theorem (this follows from the fact that it is asymptotically holomorphic). It implies that whenever the dimension of M is greater than 3, the morphism

$$H_0(B) \longrightarrow H_0(M)$$

is surjective. Hence we conclude that B is not the empty set. $\square$

The triviality of the Chern class of the quasi–contact structures on the fibres and the non–emptiness of B are used in the construction of the contact structure.

4. Base locus and Critical loops.

Let (M, ξ, ω) be an exact quasi–contact 5–manifold and (f, B, C) a quasi–contact pencil on it. Assume that $B \neq \emptyset$ and $c_1(\xi_F) = 0$ for a regular fibre F of f. Such a pencil is provided in Corollary 3.6. A fair amount of control on the almost–contact structure can be achieved in the neighborhood of the base locus and the critical loops.

DEFINITION 4.1. A submanifold $i : S \longrightarrow M$ of an almost contact manifold (M, ξ, ω) is said to be contact if it is an almost contact submanifold and there is a choice of adapted form α for ξ in a neighborhood U of S, i.e. $\xi_{|U} = \ker \alpha$, such that $(d\alpha)_{|U} = \omega_{|U}$.

An additional property in our almost contact pencil can then be required.

DEFINITION 4.2. An almost contact pencil (f, B, C) on (M, ξ, ω) is called good if $B \neq \emptyset$, any smooth fibre F satisfies $c_1(\xi_F) = 0$ and B and C are contact submanifolds of (M, ξ, ω).

The following lemma provides a perturbation achieving a suitable almost contact pencil.

LEMMA 4.3. *Let (M, ξ, ω) be a quasi–contact closed 5–dimensional manifold and let (f, B, C) be a quasi–contact pencil. There exists a C^0–small perturbation $\{(\xi_t, \omega)\}$ of almost contact structures such that:*

(1) (ξ_t, ω) is an almost contact structure $\forall t \in [0, 1]$,
and the almost contact structure (ξ_0, ω) equals (ξ, ω).
(2) B and C are contact submanifolds of (ξ_1, ω).
(3) (f, B, C) is an almost contact pencil for (M, ξ_1, ω).
(4) $c_1((\xi_1)_{|F}) = 0$ for any regular fibre F of f.

Fix an associated contact form α, i.e. $\xi = \ker \alpha$. The proof of the lemma is an exercise. Indeed, in a neighborhood of the link $B \cup C$ the difference between ω and $d\alpha$ is exact and its primitive (which can be chosen to vanish along the link) allows us to perturb the defining form until we achieve the contact condition $\omega = d\alpha_1$, $\xi_1 = \ker \alpha_1$.

Both Corollary 3.6 and Lemma 4.3 imply the following

PROPOSITION 4.4. *Let (M, ξ, ω) be an exact quasi–contact closed 5–dimensional manifold. Then there exists an almost contact perturbation (ξ', ω) of (ξ, ω) such that (M, ξ', ω) admits a good almost contact pencil (f, B, C).*

5. Surgery and good ace fibrations

Let (f, B, C) be a good almost contact pencil on (M, ξ, ω). The map f does not define a smooth fibration on M for two reasons: it is not defined on B and there exist critical fibres. The former failure can be avoided if we change the domain manifold M, i.e. f can be defined on a suitable closed manifold $\widetilde{M}$ obtained from M by a specific surgery procedure. Let us introduce three pieces of terminology.

DEFINITION 5.1. An almost contact Lefschetz fibration is an almost contact pencil (f, B, C) with $B = \emptyset$. A contact Lefschetz fibration is a contact pencil (f, B, C) with $B = \emptyset$.

DEFINITION 5.2. An almost contact exceptional fibration on (M, ξ, ω) is a triple (f, C, E) where (f, C) is an almost contact Lefschetz fibration and E a non–empty collection of embedded 3–spheres with trivial normal bundle such that f restricts to the Hopf fibration on any of them.

An almost contact exceptional fibration will be shortened to an ace fibration.

DEFINITION 5.3. An ace fibration is said to be good if the curves C and the spheres in E are contact submanifolds of (M, ξ, ω), the contact structure in any 3–sphere of E is the standard tight contact structure and any smooth fibre F of f satisfies $c_1(\xi_F) = 0$.

An almost contact Lefschetz fibration can be obtained out of an almost contact Lefschetz pencil by performing a surgery along the base locus. In particular, each connected component of the link B is replaced by a standard 3–sphere $(\mathbb{S}^3, \xi_{std})$. The aim of this Section is to produce a good ace fibration from a good almost contact pencil on a 5–dimensional manifold.

THEOREM 5.1. *Let (M, ξ, ω) be an almost contact 5–manifold and (f, B, C) a good almost contact pencil. There exist a homotopic deformation (ξ_1, ω_1) of (ξ, ω), an almost contact manifold $(\widetilde{M}, \widetilde{\xi}, \widetilde{\omega})$ with a good ace fibration $(\widetilde{f}, E, \widetilde{C})$, a closed neighborhood $\mathcal{N}(B)$ of B and a diffeomorphism $\Pi : \widetilde{M} \setminus E \longrightarrow M \setminus \mathcal{N}(B)$ such that*

- *The almost contact structure (ξ_1, ω_1) is contact on a neighborhood of $\mathcal{N}(B)$.*
- $(\Pi_* \widetilde{\xi}, \Pi_* \widetilde{\omega}) = (\xi_1, \omega_1)$ *on $M \setminus \mathcal{N}(B)$.*

Note that in the context of this Chapter, we are implicitly assuming that the map f has been constructed using asymptotically holomorphic techniques and thus the map f is defined using a section of the bundle $\mathbb{C}^2 \otimes \det(\xi)$ (we refer the reader to the paragraph following Theorem 3.1). The description of the almost contact manifold $(\widetilde{M}, \widetilde{\xi}, \widetilde{\omega})$ is explicit from the data (M, ξ, ω). The good ace fibration $(\widetilde{f}, E, \widetilde{C})$ is also constructed directly from (f, B, C). This procedure we use is a particular case of a blow–up operation. The analogy with the blow–up of a base point for a symplectic Lefschetz pencil on a 4–manifold can be useful for the reader. See also Chapter 4 in this dissertation.

The description of $(\widetilde{M}, \widetilde{\xi}, \widetilde{\omega})$ is given in Section 5.1. The compatibility of $(\widetilde{M}, \widetilde{\xi}, \widetilde{\omega})$ with the fibration $(\widetilde{f}, C)$ is detailed in Subsection 5.2. In Subsection 5.3, we describe a method that ensures that the regular fibres of the new fibration $\widetilde{f}$ have vanishing Chern class.

5.1. Surgery. The almost contact manifold $(\widetilde{M}, \widetilde{\xi}, \widetilde{\omega})$ is obtained from (M, ξ, ω) via a surgery procedure. The only topological requirement to perform surgery along a sphere is the triviality of its normal bundle.

In contact topology, a standard contact neighborhood also appears in the description. In particular there exists a restriction on the radius in the local model. See [**107**]. This is not an issue in the almost contact case: the size of a neighborhood of a contact submanifold of an almost contact manifold can be enlarged by a homotopy of the distribution. In precise terms:

LEMMA 5.4. *Let (M, ξ, ω) be an almost contact manifold and $(S, \xi = \ker \alpha)$ be a contact submanifold with trivial normal bundle $\nu_S \cong S \times \mathbb{R}^{2q}$. Fix a radius $R \in \mathbb{R}$. Then there exists an almost contact homotopy (M, ξ_t, ω_t) such that $(M, \xi_0, \omega_0) = (M, \xi, \omega)$ and it conforms the following conditions:*

- *The homotopy is supported in an annulus around S, i.e. given a smooth fiberwise metric on ν_S there exist $\rho_1, \rho_2 \in \mathbb{R}^+$ with $\rho_1 < \rho_2$ such that*

$$\xi_t|_{\mathbb{D}(\nu_S,\rho_1)} = \xi|_{\mathbb{D}(\nu_S,\rho_1)}, \quad \xi_t|_{M\setminus\mathbb{D}(\nu_S,\rho_2)} = \xi|_{M\setminus\mathbb{D}(\nu_S,\rho_2)},$$

  where $\mathbb{D}(\nu_S, r)$ is the disk bundle of radius r. The almost contact homotopy can be chosen such that ρ_1, ρ_2 are arbitrarily small.

- *There exist a neighborhood U of S and a diffeomorphism φ such that*

$$\varphi : S \times B^{2q}(R) \longrightarrow U, \quad \varphi^*\xi_1 = \ker(\alpha - r^2\alpha_{std}), \quad \varphi^*\omega_1 = d\alpha - 2rdr \wedge d\alpha_{std},$$

  where the 1–form α_{std} is the standard contact form on $\partial B^{2q}(R)$.

PROOF. This is a statement about a neighborhood $S \times B^{2q}(\varepsilon)$. Suppose that $R > \varepsilon$. In $S \times B^{2n}(\varepsilon)$ the almost contact distribution (ξ, ω) is a contact structure described as the kernel of the 1–form $\eta_0 = \alpha - r^2\alpha_{std}$. Consider a function $H \in C^\infty([0, \varepsilon], \mathbb{R}^+)$ such that:

a. $H(r) = r^2$ for $r \in [0, \varepsilon/4] \cup [3\varepsilon/4, \varepsilon]$,
b. $H'(r) > 0$ for $r \in (0, \varepsilon/2)$,
c. $H(\varepsilon/2) = R^2$.

Consider the two values $\rho_1 = \varepsilon/4$ and $\rho_2 = \varepsilon$. There exists a homotopy $\{H_t\}$ of functions in $C^\infty([0, \varepsilon], \mathbb{R}^+)$ with $H_0(r) = r^2$, $H_1(r) = H(r)$ and any H_t satisfying properties a and b above. The homotopy of 1–forms $\eta_t = \alpha - H_t(r)\alpha_{std}$ defines a homotopy of almost contact distributions. The distributions are $\xi_t = \ker \eta_t$. The symplectic structures are of the form $\omega_t = d\alpha - H_t d\alpha_{std} - \mathcal{H}_t(r)dr \wedge \alpha_{std}$ where $\mathcal{H}_t(r)$ is a positive smooth function coinciding with $\partial_r H_t$ in $r \in [0, \varepsilon/2) \cup (3\varepsilon/4, \varepsilon]$. The diffeomorphism

$$\begin{array}{rcl} \Psi : S \times B^{2q}(R) & \longrightarrow & S \times B^{2q}(\varepsilon/2) \\ (s, r, \theta) & \longmapsto & (s, \sqrt{H(r)}, \theta) \end{array}$$

satisfies $\Psi^*\eta_0 = \eta_1$ and the statement of the Lemma follows. $\square$

The Lemma does not hold for a contact structure since the contact condition is violated at the region $(\varepsilon/2, 3\varepsilon/4)$ in the course of the homotopy.

Theorem 5.1 concerns both the construction of an almost contact manifold and a good ace fibration. The description of the former naturally leads to that of the latter. Let us then begin with the almost contact manifold. Both the statement and the proof of the following result are relevant. Subsections 5.2 and 5.3 refer to the proof and notation therein.

THEOREM 5.2. *Let (M^{2n+1}, ξ, ω) be an almost contact manifold and $S \subset M$ a smooth transverse loop. Suppose that (ξ, ω) is a contact structure on a neighborhood of S. There exist a homotopic deformation (ξ_1, ω_1) of (ξ, ω), a manifold $\widetilde{M}$, a codimension–2 submanifold $E \subset \widetilde{M}$, a neighborhood $\mathcal{N}(S)$ of S and a diffeomorphism $\Pi : \widetilde{M} \setminus E \longrightarrow M \setminus \mathcal{N}(S)$ conforming the following conditions:*

- *There exists an almost contact structure $(\widetilde{\xi}, \widetilde{\omega})$ on $\widetilde{M}$.*
- *The codimension–2 submanifold E is a contact submanifold of $(\widetilde{M}, \widetilde{\xi}, \widetilde{\omega})$ contactomorphic to the standard contact sphere $(\mathbb{S}^{2n-1}, \xi_{st})$.*
- *$(\Pi_*\widetilde{\xi}, \Pi_*\widetilde{\omega}) = (\xi_1, \omega_1)$ on $M \setminus \mathcal{N}(S)$.*

The submanifold E is called the exceptional divisor.

PROOF. This proof depends on a fixed integer $k \in \mathbb{Z}$. This parameter becomes relevant in the description of the good ace fibration $(\widetilde{f}, E, \widetilde{C})$. It can be chosen quite arbitrarily in this argument, but there shall be a specific choice in the proof of Theorem 5.1.

Consider the standard contact form α_{std} on $\mathbb{S}^{2n-1}$, induced by the restriction of the standard Liouville form on $\mathbb{R}^{2n}$, and the contact structure $\xi_{std} = \ker\{d\theta - \rho^2\alpha_{std}\}$ on $\mathbb{S}^1 \times B^{2n}$ endowed with polar coordinates $(\theta; \rho, \sigma)$. The contact neighborhood theorem for the transverse loop S

provides an open neighborhood U of S, a constant $\rho_0 \in \mathbb{R}^+$ and a diffeomorphism

$$\begin{aligned} \phi : S \times B^{2n}(\rho_0) &\longrightarrow U \\ (\theta, \rho, \sigma) &\longmapsto \phi(\theta, \rho, \sigma) \end{aligned}$$

such that $\phi^*(\xi_{|U}) = \xi_{std}$. If k is a positive integer, suppose that the radius ρ_0 is small enough so that $k\rho_0^2 < 1$. This condition is necessarily satisfied for $k < 0$. Consider the positive number $\rho_k \in \mathbb{R}^+$ satisfying $\rho_0 = \frac{\rho_k}{\sqrt{1+k\rho_k^2}}$ and the diffeomorphism

$$\begin{aligned} \psi_k : \mathbb{S}^1 \times B^{2n}(\rho_k) &\longrightarrow \mathbb{S}^1 \times B^{2n}(\rho_0) \\ (\theta, \rho, w_1, \ldots, w_n) &\longmapsto \left(\theta, \frac{\rho}{\sqrt{1+k\rho^2}}, e^{ik\theta}w_1, \ldots, e^{ik\theta}w_n\right). \end{aligned}$$

The map ψ_k preserves the distribution ξ_{std}. In case it is needed, apply the Lemma 5.4 to enlarge the neighborhood $\mathbb{S}^1 \times B^{2n}(\rho_k)$ of S to radius $R = 2$. This yields a deformation ξ_1 of the contact structure ξ_{std} supported in an annulus of radii $0 < \rho_a < \rho_b < \rho_k$ and a compatible embedding $\varphi : \mathbb{S}^1 \times B^{2n}(2) \longrightarrow \mathbb{S}^1 \times B^{2n}(\rho_b)$. The deformation is relative to the boundary and thus the distribution $(\phi \circ \psi_k \circ \varphi)_*(\xi_1)$ defined over U admits an extension ξ_1 over M using the original distribution ξ. There is also a corresponding extension for the symplectic structure ω_1. To ease notation, we still refer to (ξ_1, ω_1) as (ξ, ω). In these terms, Lemma 5.4 provides a neighborhood U' of S in M and a diffeomorphism

$$\Phi : \mathbb{S}^1 \times B^{2n}(2) \longrightarrow U', \quad (\theta, r, \sigma) \longmapsto \Phi(\theta, r, \sigma) = \phi \circ \psi_k \circ \varphi,$$

$$\text{such that } \Phi^*(\xi|_S) = \ker(d\theta - r^2\alpha_{std}).$$

Consider the diffeomorphism

$$\begin{aligned} \phi_1 : \mathbb{S}^1 \times (3/2, 2) \times \mathbb{S}^{2n-1} &\longrightarrow \mathbb{S}^1 \times (3/2, 2) \times \mathbb{S}^{2n-1} \\ (\theta, r, w_1, \ldots, w_n) &\longrightarrow (\theta, r, e^{i\theta}w_1, \ldots, e^{i\theta}w_n). \end{aligned}$$

If $V = \Phi(\mathbb{S}^1 \times B^{2n}(3/2))$, then the map

$$g = \Phi \circ \phi_1 : \mathbb{S}^1 \times (3/2, 2) \times \mathbb{S}^{2n-1} \longrightarrow U \setminus V \subset M$$

satisfies

$$g^*\xi = \ker\left\{-\left(\alpha_{std} + \frac{r^2-1}{r^2}d\theta\right)\right\}.$$

Note that the function

$$\begin{aligned} h : (3/2, 2) &\longrightarrow \mathbb{R} \\ r &\longmapsto h(r) = \frac{r^2 - 1}{r^2} \end{aligned}$$

satisfies $h(r) > 5/9$. Therefore it is possible to extend it to a smooth function $\widetilde{h} : [0, 2) \longrightarrow \mathbb{R}$ satisfying the following conditions (See Figure 1):

- $\widetilde{h}(r) = r^2$, for $r \in [0, 1/2]$,
- $\widetilde{h}(r) = h(r)$, for $r > 3/2$,
- $\widetilde{h}(r)' > 0$ for $r \in [1/2, 3/2]$.

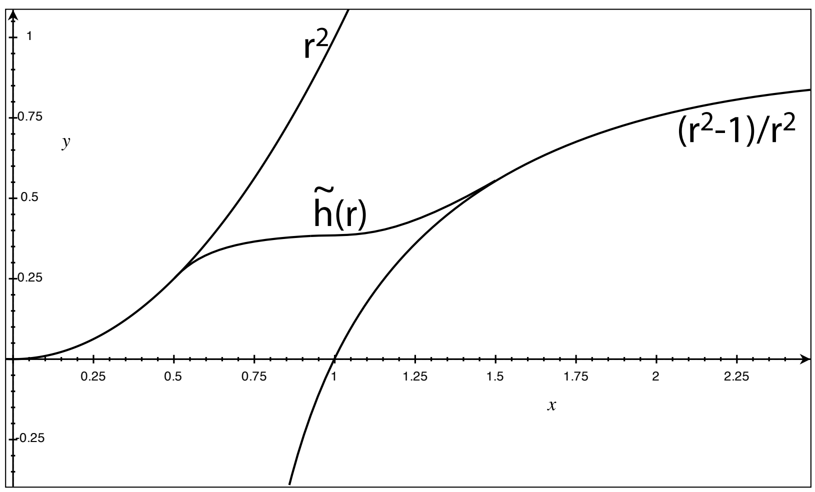


FIGURE 3. The function $\widetilde{h}$.

Therefore $\widetilde{\eta} = -\alpha_{std} - \widetilde{h}(r)d\theta$ defines a distribution $\widetilde{\xi}$ over $\mathbb{S}^1 \times [0, 2) \times \mathbb{S}^{2n-1} \cong B^2(2) \times \mathbb{S}^{2n-1}$. Note that $\widetilde{\eta}$ is a contact form near the core $\{0\} \times \mathbb{S}^{2n-1}$. We can glue the manifold $(M \backslash V, \xi)$ and $(B^2(2) \times \mathbb{S}^{2n-1}, \ker \widetilde{\eta})$ with the gluing map g to define an almost contact manifold $(\widetilde{M}, \widetilde{\xi})$. This manifold satisfies the statement of the theorem with

$$\mathcal{N}(S) = \Phi(\mathbb{S}^1 \times B^{2n}(1)).$$

$\square$

5.2. Compatibility with an almost contact pencil. Let (f, B, C) be a good almost contact pencil on a 5–dimensional almost contact manifold (M, ξ, ω). The almost contact structure (ξ_1, ω_1) obtained in

Lemma 5.4 can be chosen to remain adapted to the almost contact pencil (f, B, C) (this can be done by proving a standard neighborhood theorem using the local models provided by the definition of a good almost contact pencil). Let us understand the choices involved in the Theorem 4.1. The map f pulls–back to

$$f \circ \Pi : \widetilde{M} \setminus E \longrightarrow \mathbb{CP}^1.$$

Due to the surgery procedure it can be extended to a map $\widetilde{f} : \widetilde{M} \longrightarrow \mathbb{CP}^1$. Let us explain this.

The first choice in the previous construction is the chart map $\phi : \mathbb{S}^1 \times B^{2n}(\rho_0) \longrightarrow U$ for a neighborhood U of a connected component $\gamma \cong \mathbb{S}^1$ in the base locus B. This amounts to a choice of framing of the trivial normal bundle along this $\mathbb{S}^1$. Since $\mathbb{S}^1 \subset B$ we can use the adapted charts in Definition 3.2 and require that ϕ satisfies that the map

$$f \circ \phi : \mathbb{S}^1 \times (B^4(\rho_0) \backslash \{0\}) \longrightarrow \mathbb{CP}^1$$

is precisely $(f \circ \phi)(\theta, w_1, w_2) = [w_1 : w_2]$. Therefore, the compactified fibres are of the form $\mathbb{S}^1 \times L$, for any complex line $L \subset \mathbb{C}^2$. It is also satisfied that $(f \circ \phi \circ \psi_k)(\theta, w_1, w_2) = [w_1 : w_2]$ and again the same compactification for the fibres still holds. Moreover the fibres are almost contact. It is left to study the effect of φ and ϕ_1.

The deformation performed in the enlargement of the neighborhood from (ξ_0, ω_0) to (ξ_1, ω_1) preserves the fibres as almost contact submanifolds. The reason being that in Lemma 5.4 the fibres in the coordinates $(\theta, \rho, \sigma) = (\theta, \rho, w_1, w_2)$ are given by the equation

$$F_z = \{(\theta, \rho, w_1, w_2) : [w_1 : w_2] = z\} \quad \text{for } z \in \mathbb{CP}^1,$$

and the restriction of (ξ_1, ω_1) is given by

$$(\ker\{d\theta + H(\rho)(\alpha_{std})_{|_{\mathbb{S}^3 \cap L_z}}\}, \mathcal{H}(\rho) d\rho \wedge (\alpha_{std})_{|_{\mathbb{S}^3 \cap L_z}}),$$

where L_z is the line represented by $z \in \mathbb{CP}^1$ and $\mathcal{H}$ is a smooth function which equals $\partial_\rho H$ in the region of radius $\rho \in [0, \rho_a) \cup (\rho_b, \rho_k]$ and it is strictly positive for $\rho \in [\rho_a, \rho_b]$. In particular, $\mathcal{H}$ is positive and the restriction of ω_1 is indeed a symplectic structure.

Let us focus on the compactification of fibres in $\widetilde{M}$, i.e. the extension of $\widetilde{f}$ from $\Pi^{-1}(M \backslash \mathcal{N}(B))$ to $\widetilde{M}$. We first restrict ourselves to the transition region $\mathbb{S}^1 \times (3/2, 2) \times \mathbb{S}^3 \subset \mathbb{S}^1 \times \mathbb{C}^2$. The gluing map is $\phi \circ \psi_k \circ \varphi \circ \phi_1$.

In order to understand the fibres we just need to describe the map $\widetilde{f} = f \circ g = f \circ \phi \circ \psi_k \circ \varphi \circ \phi_1$. We can easily verify that

$$\widetilde{f}(\theta, r, w_1, w_2) = (f \circ g)(\theta, rw_1, rw_2) = [w_1 : w_2]$$

since $\phi \circ \psi_k \circ \varphi$ and ϕ_1 act as complex scalar multiplication in the transition area.

Notice that the domain of definition of $\widetilde{f}$ is $\mathbb{S}^1 \times (3/2, 2) \times \mathbb{S}^3$, and it is invariant with respect to the coordinates $(\theta, r) \in \mathbb{S}^1 \times (3/2, 2)$. Hence, the map $\widetilde{f}$ extends trivially to the model $(B^2(2) \times \mathbb{S}^3, \ker \widetilde{\eta})$. In particular, the extension of $\widetilde{f}$ restricted to the exceptional divisor $\{0\} \times \mathbb{S}^3$ is the Hopf fibration.

The fibres of the fibration $\widetilde{f}$ are thus almost contact submanifolds. The critical locus $\widetilde{C}$ is in bijection with C and it is a contact submanifold since the almost contact structure remains unchanged near them. The exceptional divisors E are also contact submanifolds and the fibres of $\widetilde{f}$ restricted to $(B^2(2) \times \mathbb{S}^3, \ker \widetilde{\eta})$ are diffeomorphic to $B^2(2) \times \mathbb{S}^1$, the $\mathbb{S}^1$–factor being a transverse Hopf fibre. These fibres are also contact submanifolds.

5.3. The good ace fibration. The fibres $\widetilde{F}$ of the Lefschetz fibration $(\widetilde{f}, \widetilde{C})$ differ from the fibres F of (f, B, C). Let us provide a precise description of $\widetilde{F}$ and show that the procedure described in the previous two subsections can be performed to obtain $c_1(\widetilde{\xi}_{\widetilde{F}}) = 0$. This concludes Theorem 5.1.

The trivialization of a neighborhood of a connected component $\gamma \cong \mathbb{S}^1 \subset B$ of the base locus provided in Definition 3.2 induces a natural framing $\nu_S \cong \mathbb{S}^1 \times \mathbb{C}^2$, i.e. $\langle (1,0), (i,0), (0,1), (0,i) \rangle$. It restricts to a framing inside the two fibres corresponding to the two complex axes of $\mathbb{C}^2$. Hence it induces framings in any complex line $\mathbb{S}^1 \times \mathbb{C} \subset \mathbb{S}^1 \times \mathbb{C}^2$: for the complex line $\{(z, w) \in \mathbb{C}^2 : z - \alpha w = 0\}$, we use $\langle (\alpha, 1), i(\alpha, 1) \rangle$. Denote by $\mathbb{F}_p(0)$ such framing of $B \subset \overline{f^{-1}(p)}$. Let $\mathbb{F}_p(n)$ be the n–twist of $\mathbb{F}_p(0)$ and k_γ be the parameter used in the construction of Theorem 4.1 when performing the surgery along γ.

LEMMA 5.5. *Let (M, ξ, ω) be an almost contact 5–manifold, (f, B, C) a good almost contact pencil adapted to it and $(\widetilde{M}, \widetilde{\xi}, \widetilde{\omega})$ a manifold as described in Theorem 4.1. Then $(\widetilde{M}, \widetilde{\xi}, \widetilde{\omega})$ has an almost contact fibration*

$(\tilde{f}, \widetilde{C})$ that coincides with (f, B, C) away from $B = \gamma_1 \cup \ldots \cup \gamma_s$. Near $\gamma \in B$ the fibre over $p \in \mathbb{CP}^1$ is contactomorphic to a transverse contact $(0,1)$–surgery performed on $\overline{f^{-1}(p)}$ along γ_i with framing $\mathbb{F}_p(-k_i - 1)$, for some $k_i \in \mathbb{Z}$. The restriction of the map f to each of the exceptional divisors is given by the Hopf fibration.

Proof. The map ψ_k in Theorem 4.1 modifies the initial framing from $\mathbb{F}_p$ to $\mathbb{F}_p(-k_i)$, $k_i = k_{\gamma_i}$ being the corresponding parameter k in the surgery along γ_i. Using the map ϕ_1 substracts another twist and sends the meridian to the longitude of the added solid torus. It is thus a $(p,q) = (0,1)$–Dehn surgery with respect to $\mathbb{F}_p(-k_i - 1)$. □

Note that the coefficients k_i can be arbitrarily chosen. The constructive argument will use the fact that $c_1(\widetilde{\xi}_{\widetilde{F}}) = 0$ for any fibre $\widetilde{F}$ of $\widetilde{f}$. This has been achieved for the initial fibres of the pencil. The procedure changes the almost contact manifold (F, ξ) to $(\widetilde{F}, \widetilde{\xi})$ and we cannot directly assume that $c_1(\widetilde{\xi}_{\widetilde{F}}) = 0$. This will be fixed in the following discussion.

Proposition 5.6. *Let (M, ξ, ω) be an almost contact 5–manifold, (f, B, C) a good almost contact pencil adapted to it and $(\widetilde{M}, \widetilde{\xi}, \widetilde{\omega})$ a manifold obtained as in Theorem 4.1. Suppose that (f, B, C) is obtained via asymptotically holomorphic sections as in Corollary 3.6. There is a choice of $(k_1, \ldots, k_s) \in \mathbb{Z}^s$ such that the first Chern class of the almost contact structure $(\widetilde{M}, \widetilde{\xi}, \widetilde{\omega})$ on any regular fibre $\widetilde{F}$ is zero.*

In the proof there is no need for the sections to be asymptotically holomorphic. The only requirement is that the pencil is obtained as the linear system associated to two sections.

Proof. Consider a connected component $\gamma \subset B$. The good almost contact pencil is obtained from a section

$$s = (s_0, s_1) : M \longrightarrow \mathbb{C}^2 \otimes \det(\xi).$$

and it is the input of Corollary 3.6.

Suppose that the section (s_0, s_1) can be lifted to a non–vanishing section $(\widetilde{s}_0, \widetilde{s}_1)$ from the manifold $\widetilde{M}$ to the bundle $\mathbb{C}^2 \otimes \det \widetilde{\xi}$. That is, the map $\tilde{f}$ comes as a quotient of two sections $(\widetilde{s}_0, \widetilde{s}_1)$ of the bundle $\det \widetilde{\xi}$. Then Lemma 3.5 implies that its regular fibres satisfy the required property. Hence, we just need to find a non–vanishing lift of the two sections (s_0, s_1). Let us show that this lift exists for a particular choice of integers

$(k_1, \ldots, k_s)$.

The study of sections of a complex bundle $\det \xi$ with $\xi \subset TM$ does not depend on the homotopy class of ξ as a complex subbundle of TM. In particular, we can deform ξ to a complex subbundle ξ_h and study the extension properties of two sections of $\det(\xi_h)$ corresponding to a deformation of (s_0, s_1). The bundle ξ_h yields simpler computations. A word of caution, the notation ξ_h will now be used to refer to a distribution in a local chart and not in the manifold M itself.

Consider polar coordinates $(\theta; r, \sigma) \in \mathbb{S}^1 \times B^4(2)$. The pull–back of the distribution ξ by the map $\Phi = \phi \circ \psi_k \circ \varphi$ is

$$\Phi^*(\xi) = \ker \eta, \quad \eta = d\theta + r^2 \alpha_{std}.$$

Let $\chi : [0, 2] \longrightarrow [0, 1]$ be a smooth increasing function such that

$$\chi|_{[0,1.7]} = 0 \text{ and } \chi|_{[1.9,2]} = 1.$$

Define the form $\eta_h = d\theta + \chi(r) r^2 \alpha_{std}$ and the distribution $\xi_h = \ker \eta_h$. The distribution $\Phi_* \xi_h$ can be extended to the manifold M using ξ. A linear interpolation between η and η_h induces a homotopy between the two complex bundles $\Phi^* \xi$ and ξ_h. The map ϕ_1 is a diffeomorphism in $\mathbb{S}^1 \times (1.5, 2) \times \mathbb{S}^3$. The pull–backs of the kernels of these two forms via the map $\phi_1|_{\mathbb{S}^1 \times (1.5,1.7) \times \mathbb{S}^3}$ are two distributions $\phi_1^*(\ker \eta)$ and $\phi_1^*(\ker \eta_h)$.

Consider the function $\widetilde{h}$ defined in the proof of Theorem 4.1 and a smooth increasing function $\sigma : [0, \infty) \longrightarrow [0, \pi/2]$ constant equal to 0 in $[0, 1/2]$ and constant equal to $\pi/2$ in $[1.5, \infty)$. Define also the form

$$\widetilde{\eta}_h = \sin(\sigma(r)) d\theta + \cos(\sigma(r)) \alpha_{std}.$$

First, the kernel of the contact form $\widetilde{\eta} = \alpha_{std} + \widetilde{h} d\theta$ extends the distribution $\phi_1^*(\ker \eta)$ to $B^2(1.7) \times \mathbb{S}^3$, with polar coordinates $(r, \theta) \in B^2(1.7)$. Let $\widetilde{\xi}$ be the push–foward to the manifold of $\ker \widetilde{\eta}$ extended by $\phi_1^*(\ker \eta)$. Second, the distribution $\phi_1^*(\ker \eta_h)$ coincides with $\ker d\theta$ in $\mathbb{S}^1 \times (1.5, 1.7) \times \mathbb{S}^3$ and $\ker \widetilde{\eta}_h$ extends $\phi_1^*(\ker \eta_h)$ to $B^2(1.7) \times \mathbb{S}^3$. Let $\widetilde{\xi}_h$ be the push–foward to the manifold of $\ker \widetilde{\eta}_h$ extended by $\phi_1^*(\ker \eta_h)$. The distributions $\ker \widetilde{\eta}$ and $\ker \widetilde{\eta}_h$ are homotopic via linear interpolation. The homotopy coincides with the homotopy between $\Phi^* \xi$ and ξ_h in the region $\mathbb{S}^1 \times (1.5, 1.7) \times \mathbb{S}^3$. Hence, the homotopy extends to a homotopy between $\widetilde{\xi}$ and $\widetilde{\xi}_h$ inside the manifold M.

Let $X_r = \partial_r, X_i = iX_r, X_j = jX_r, X_k = kX_r$ be a basis generating $T\mathbb{C}^2 = \mathbb{C}^2 \cong \mathbb{H}^1$. Consider the chart defined by ϕ with polar coordinates

$$(\theta; r, w_0, w_1) \in \mathbb{S}^1 \times \mathbb{C}^2 \cong \mathbb{S}^1 \times \mathbb{R}^{\geq 0} \times \mathbb{S}^3.$$

The distribution $\xi_h = \ker d\theta$ will be identified with $\mathbb{C}^2$. The original sections (s_0, s_1) will be identified as sections of $\Phi_* \det \xi_h$. Suppose the sections (s_0, s_1) restrict to an m–twisted frame, i.e. in the chart above the pair of sections is written up to homotopy as

$$\phi^*(s_0, s_1) \simeq e^{m \cdot i\theta}(w_0, w_1)(1,0) \wedge (0,1).$$

The change of coordinates is defined, up to homotopy, by

$$(\psi_k \circ \varphi \circ \phi_1)(\theta, r, w_0, w_1) = (\theta, r, e^{i(1+k)\theta} w_0, e^{i(1+k)\theta} w_1).$$

It pulls–back the basis framing to

$$(\psi_k \circ \varphi \circ \phi_1)^*(1,0) \wedge (0,1) = e^{-2i(1+k)\theta}(1,0) \wedge (0,1).$$

Therefore the pull–back of the 2 sections is

$$\begin{aligned}(\Phi \circ \phi_1)^*(s_0, s_1) &= (\phi \circ \psi_k \circ \varphi \circ \phi_1)^*(s_0, s_1) \\ &\simeq e^{(m-k-1)\cdot i\theta}(w_0, w_1)(1,0) \wedge (0,1) \\ &= e^{(m-k-1)\cdot i\theta}(w_0, w_1) X_r \wedge X_j \\ &= -ie^{(m-k-1)\cdot i\theta}(w_0, w_1) X_i \wedge X_j.\end{aligned}$$

Observe that k controls the twisting of the section around the component γ. The distribution ξ_h is extended to $B^2(1.7) \times \mathbb{S}^3$ with the distribution $\Phi^*\widetilde{\xi}_h$. The four vector fields X_r, X_i, X_j, X_k define a framing of ξ_h in $\mathbb{S}^1 \times (1.5, 1.7) \times \mathbb{S}^3$. This framing needs to be extended to the interior $B^2(1.7) \times \mathbb{S}^3$ to a framing of the distribution

$$\ker \widetilde{\eta}_h = \ker\{\sin(\sigma(r))d\theta + \cos(\sigma(r))\alpha_{std}\}.$$

A possible extension is given by $\langle X_r, \sin(\sigma(r))X_i - \cos(\sigma(r))\partial_\theta, X_j, X_k\rangle$.

Consider $p = m - k - 1$ and let us identify $\Phi_*\xi_h$ and $\widetilde{\xi}_h$ in their common region. The section $(\Phi \circ \phi_1)^*(s_0, s_1)$ seen as a section of $\mathbb{C}^2 \otimes \det \widetilde{\xi}_h$ can be extended to

$$(\widetilde{s}_0, \widetilde{s}_1) \simeq -ie^{p\cdot i\theta}(w_0, w_1)(\sin(\sigma(r))X_i - \cos(\sigma(r)) \cdot \partial_\theta) \wedge X_j.$$

Thus it is an extension of the section to $\widetilde{M}$. For radius $r = 0$, in the new compactification $B^2(r, \theta) \times \mathbb{S}^3(w_0, w_1)$, the section reads

$$(\widetilde{s}_0, \widetilde{s}_1) = ie^{p\cdot i\theta}(w_0, w_1)\partial_\theta \wedge X_j,$$

which extends without zeroes if and only if $p = -1$. The choice $k = m$ allows us to extend the section $(\widetilde{s}_0, \widetilde{s}_1)$ to the interior of the exceptional sphere without zeroes.

In short, the required section $\widetilde{s} = (\widetilde{s}_0, \widetilde{s}_1)$ extends to the previous section $s = (s_0, s_1)$ away from the surgery area. Since the sections can be extended to the manifold $\widetilde{M}$ in a non–vanishing manner we conclude $c_1(\widetilde{\xi}|_{\widetilde{F}}) = 0$ and the base locus is empty, that is $\widetilde{B} = \emptyset$. $\square$

This concludes the proof of Theorem 5.1. The argument developed in this Chapter to prove Theorem 1.1 requires a smooth fibration, hence the reason for Theorem 5.1. There is an alternative approach not involving the manifold $\widetilde{M}$ that leads to a quite complicated version of the local models used in Sections 6, 7 and 8. These models are essential to describe the deformation of the almost contact structure. The simpler, the better. In particular, the description in Section 8 would be rather technical if the modified model was used.

6. Vertical Deformation.

In Section 3 we endowed our initial 5–dimensional almost contact manifold (M, ξ, ω) with an almost contact pencil (f, B, C) such that $B \neq 0$ and $c_1(\xi_F) = 0$ for the fibres F of f. In Proposition 5.6 we have obtained a contact structure in a neighborhood of the base locus B and the critical curves C. According to Theorem 5.1 there exists a good ace fibration $(\widetilde{f}, E, \widetilde{C})$ in an almost contact manifold $(\widetilde{M}, \widetilde{\xi}, \widetilde{\omega})$ isomorphic to $(M \setminus \mathcal{N}(B), \xi, \omega)$ away from a codimension–2 contact submanifold E. In order to obtain a contact structure in the manifold (M, ξ, ω) we use the splitting induced by the existence of the Lefschetz fibration $(\widetilde{f}, \widetilde{C})$ on $(\widetilde{M}, \widetilde{\xi}, \widetilde{\omega})$. Henceforth we shall consider an almost contact manifold with a good ace fibration. These will be respectively denoted (M, ξ, ω) and (f, C, E) even though in our situation they refer to the manifold $(\widetilde{M}, \widetilde{\xi}, \widetilde{\omega})$ and the good ace fibration $(\widetilde{f}, \widetilde{C}, E)$. This should not lead to confusion. The initial manifold is recovered in Section 9.

Let (M, ξ, ω) be a 5–dimensional closed orientable almost contact manifold.

DEFINITION 6.1. An almost contact structure (M,ξ,ω) is called vertical contact with respect to an almost contact fibration (f,C) if the fibres of f are contact submanifolds for (ξ,ω) away from the critical points.

The main result of this section reads:

THEOREM 6.1. *Let (M,ξ,ω) be an almost contact manifold and (f,C,E) an associated good ace fibration. Then there exists a homotopic deformation of the almost contact structure relative to C and E such that the almost contact structure becomes vertical contact for (f,C).*

The proof of the theorem relies on the existence of an overtwisted disk in each fibre, such structure allows more flexibility in handling families of distributions. Hence, it will be essential for the argument to apply that the fibres of the good ace fibration (f,C,E) are 3–dimensional manifolds. In order to obtain a vertical contact fibration we need Eliashberg's classification result of overtwisted contact structures [**45**].

The almost contact structure obtained in Theorem 6.1 is constructed as a deformation of the vertical distributions $\{\xi_z = \xi \cap Tf^{-1}(z)\}_{z\in\mathbb{CP}^1}$ relative to open neighborhoods of C and E. A naive description of the argument consists of two parts. An overtwisted disk is first introduced in each fibre. This is the content of Subsection 6.2. Then Eliashberg's result allows us to deform the family $\{\xi_z\}_{z\in\mathbb{CP}^1}$ to a family of overtwisted contact structures. This corresponds to Subsection 6.3.

This argument cannot be readily applied because of two issues. On the one hand the almost contact fibration does not necessarily admit a section. In particular there is no naturally prescribed continuous family of overtwisted disks. This is solved using two local families to deal with each of the fibres. On the other hand the argument in [**45**] deals with families of distributions over a fixed manifold. In our case the topology of the fibres changes if a curve in $f(C)$ is crossed. Therefore a refined version of Eliashberg's arguments is needed. It strongly uses the relative character of the result, both with respect to the parameter spaces and the open subsets of the manifold.

A technical step requires to define a suitable finite open cover of $\mathbb{CP}^1$ by 2–disks. In particular, the fibres over each 2–disk are diffeomorphic relative to a certain subset and there exists a continuous choice of overtwisted disks over each of these fibres. This cover is associated to (f, C) and a cell decomposition of $\mathbb{CP}^1$. This will be explained.

6.1. 3–dimensional Overtwisted Structures. Our setup provides a fibration with a distribution on each fibre. Given such an almost contact fibration $f : M \longrightarrow \mathbb{CP}^1$, let F_z denote the fibre over $z \in \mathbb{CP}^1$ and (ξ_z, ω_z) the induced almost contact structure on F_z. Then the family (F_z, ξ_z) can locally be viewed as a 2–parametric family of 2–distributions on a fixed fibre.

In the proof of Theorem 6.1 we use a relative version of the following:

THEOREM 6.2 (Theorem 3.1.1 in [**45**]). *Let M be a compact closed 3–manifold and let G be a closed subset such that $M \setminus G$ is connected. Let K be a compact space and L a closed subspace of K. Let $\{\xi_t\}_{t\in K}$ be a family of cooriented 2–plane distributions on M which are contact everywhere for $t \in L$ and are contact near G for $t \in K$. Suppose there exists an embedded 2–disk $\mathcal{D} \subset M \setminus G$ such that ξ_t is contact near $\mathcal{D}$ and $(\mathcal{D}, \xi_t)$ is equivalent to the standard overtwisted disk for all $t \in K$. Then there exists a family $\{\xi'_t\}_{t\in K}$ of contact structures of M such that ξ'_t coincides with ξ_t near G for $t \in K$ and coincides with ξ_t everywhere for $t \in L$. Moreover ξ'_t can be connected with ξ_t by a homotopy through families of distributions that is fixed in $(G \times K) \cup (M \times L)$.*

In order to allow the case of a 3–manifold with non–empty boundary we also need:

COROLLARY 6.2. *Let M be a compact 3–manifold with boundary ∂M and let G be a closed subset of M such that $M \setminus G$ is connected and $\partial M \subset G$. Let K be a compact space and L a closed subspace of K. Let $\{\xi_t\}_{t\in K}$ be a family of cooriented 2–plane distributions on M which are contact everywhere for $t \in L$ and are contact near G for $t \in K$. Suppose there exists an embedded 2–disk $\mathcal{D} \subset M \setminus G$ such that ξ_t is contact near $\mathcal{D}$ and $(\mathcal{D}, \xi_t)$ is equivalent to the standard overtwisted disk for all $t \in K$. Then there exists a family $\{\xi'_t\}_{t\in K}$ of contact structures of M such that ξ'_t coincides with ξ_t near G for $t \in K$ and coincides with ξ_t everywhere for $t \in L$. Moreover ξ'_t can be connected with ξ_t by a homotopy through families of distributions that is fixed in $(G \times K) \cup (M \times L)$.*

OUTLINE. The proof for the closed case uses a suitable triangulation P of the 3–manifold having a subtriangulation Q containing G, for which the distributions are already contact structures. Then Eliashberg's argument is of a local nature, working with neighborhoods of the 0, 1, 2 and 3–skeleton of $P\backslash Q$ and assuring that no changes are made in a neighborhood of Q. Thus the method for a manifold M with $\partial M \neq 0$ is still valid since P and Q do exist in this case and only Q contains the boundary. $\square$

We locally treat an almost contact fibration as a 2–parametric family of distributions over a fixed fibre, thus we may use a disk as a parameter space and the central fibre as the fixed manifold. It will be useful to be able to obtain a continuous family of distributions such that the distributions in a neighborhood of the central fibre become contact structures while the distributions near the boundary are fixed. Such a family is provided in the following

Corollary 6.3. *Consider the notation and hypotheses of Corollary 6.2 with K diffeomorphic to a disk, $\mathcal{S} = \partial K$ its boundary sphere and coordinates $(p,r) \in \mathcal{S}\times[0,1]$. Let $\{\xi_t\}$ be a family of distributions parametrized by $\mathcal{S}\times[0,1]$ which are contact near G and $\mathcal{D}$. Suppose that $\{\xi_t\}$ are contact distributions for $t \in \lambda \subset \mathcal{S}\times[0,1]$. Given a homotopy $\xi^s_{(p,0)}$ of the distributions over $\mathcal{S}\times\{0\}$, $s\in[0,1]$, there exists a homotopy $\{\xi^s_t\}$ relative to $G\times\mathcal{S}\times[0,1]\cup M\times\lambda$ such that*

$$\xi^0_t = \xi_t, \quad \xi^s_t = \xi^s_{(p,0)} \text{ for } t=(p,0) \text{ and } \xi^1_t = \xi_t \text{ for } t=(p,1).$$

The assumption that K is a disk is not necessary. But we use Corollary 6.3 only in such a case. Its proof is left as an exercise for the reader.

We need at least one overtwisted disk over each fibre in order to apply Corollary 6.2. The family should behave continuously. Let us provide such a family of disks.

6.2. Families of overtwisted disks. There are two basics issues to be treated: the location of the disks and their overtwistedness. The second issue is simply guaranteed since once a disk with a contact neighborhood is placed in each fibre we can produce overtwisted disks using Lutz twists. In order to decide the location of the disks in each fibre we need to find a section of the good ace fibration.

Let (f,C,E) be a good ace fibration. Denote by $U(C), U(E_i)$ open neighborhoods of the critical curves C and the exceptional spheres $E_i \in E$. Consider $U(f) = U(C)\cup U(E_i)$ the union of these open neighborhoods, so in the complement of $U(f)$ the map f becomes a submersion. Instead of finding a global section mapping away from $U(f)$, we shall construct two disjoint local sections that will provide at least one overtwisted disk in each fibre $F_z = f^{-1}(z)$. The distribution $\xi_z = \xi \cap TF_z$ is well–defined over $F_z \setminus U(f)$ and varies smoothly with the parameter $z \in \mathbb{CP}^1$. The global situation we achieve is described as follows:

Proposition 6.4. *Let (f, C, E) be a good ace fibration for (M, ξ, ω). Consider two open disks $\mathcal{B}_0, \mathcal{B}_\infty \subset \mathbb{CP}^1$, containing 0 and ∞ respectively such that the intersection $\mathcal{B}_0 \cap \mathcal{B}_\infty$ is an open annulus, the complement of $\mathcal{B}_0 \cap \mathcal{B}_\infty$ consists of two disjoint disks and the curves $\partial\mathcal{B}_0, \partial\mathcal{B}_\infty$ are disjoint from the set of curves $f(C)$.*

Then there exists a deformation $(F_z, \widetilde{\xi}_z)_{z\in\mathbb{CP}^1}$ of the family $(F_z, \xi_z)_{z\in\mathbb{CP}^1}$ fixed at the intersection of the set $U(f)$ with each F_z such that there are two disjoint families of embedded 2–disks $\mathcal{D}^i_z \subset F_z$, with $z \in \mathcal{B}_i$, for $i = 0, 1$, not intersecting $U(f)$. The distribution $\widetilde{\xi}_z$ is a contact structure in a neighborhood of such families and $(\mathcal{D}^i_z, \widetilde{\xi}_z)$ are equivalent to standard overtwisted disks.

The fact that $\widetilde{\xi}_z$ equals ξ_z in the intersection of the set $U(f)$ with F_z ensures that no deformation is performed near the critical curves nor the exceptional spheres. This is mainly a global statement, involving the whole of the fibres. In order to prove the result we study the local model of a tubular neighborhood of an exceptional divisor of the good ace fibration (f, C, E).

A good ace fibration (f, C, E) is obtained by surgery along the base locus B of a certain good almost contact Lefschetz pencil. Let K_i be a knot belonging to this base locus B. After the surgery procedure it is replaced by an exceptional contact divisor $E_i \in E$ contactomorphic to $(\mathbb{S}^3, \xi_{st})$. As explained in Section 5 the restriction of the fibration f to E_i is the Hopf fibration. Since the distribution ξ is locally a contact structure the tubular neighborhood theorem provides a chart

$$\Psi : U \longrightarrow \mathbb{S}^3 \times \mathbb{D}^2(\varepsilon), \quad \Psi^*\xi_{st} = \xi \tag{6.1}$$

where $\xi_{st} = \ker\{\alpha_{\mathbb{S}^3} + r^2 d\theta\}$, $\varepsilon \in \mathbb{R}^+$ and $\Psi(E_i) = \mathbb{S}^3 \times \{0\}$. Suppose $\varepsilon = 1$ in order to ease notation.

The induced map f_U defined as

$$\begin{tikzcd} \mathbb{S}^3 \times \mathbb{D}^2 \arrow[r, "\Psi^{-1}"] \arrow[dr, "f_U"'] & U \arrow[d, "f"] \\ & \mathbb{CP}^1 \end{tikzcd}$$

can be expressed as $f_U(x, r, \theta) = h(x)$ for $x \in \mathbb{S}^3$. The fibres $F_z = f^{-1}(z) \cap U$ are contact submanifolds of $(\mathbb{S}^3 \times \mathbb{D}^2, \xi_{std})$. The induced contact structure $\xi_v(z)$ on F_z depends on the point $z \in \mathbb{CP}^1$. These fibres are contactomorphic to $(\mathbb{S}^1 \times \mathbb{D}^2, \xi_v = \ker(d\beta + r^2 d\theta))$ for each $z \in \mathbb{CP}^1$. Note that the variable $\beta \in \mathbb{S}^1$ parametrizing each Hopf fibre is not global since the fibration is not trivial. The differential $d\beta$ is globally well–defined since it is dual to the vector field generating the associated $\mathbb{S}^1$–action. The standard contact structure in $\mathbb{S}^3 \times \mathbb{D}^2$ can be expressed as the direct sum of distributions

$$\xi_{st}(x, r, \theta) = \xi_v(h(x)) \oplus H(x, r, \theta), \tag{6.2}$$

where ξ_v is the standard contact structure in $\mathbb{S}^1 \times \mathbb{D}^2$, the vertical direction, and H is a horizontal complement associated to the fibration of $\mathbb{S}^3 \times \mathbb{D}^2$ over $\mathbb{CP}^1$.

Topologically, the 4–distribution ξ_{st} is expressed as a direct sum of two distributions of 2–planes. Since the 2–form ω providing the almost contact structure is given and so is ξ, we may interpret $(\mathbb{S}^3 \times \mathbb{D}^2, \xi_v(z))$ as a non–trivial family of contact structures parametrized by the base $z \in \mathbb{CP}^1$. We have detailed the topology and contact structure of the local model of the good ace fibration along an exceptional sphere E_i. A neighborhood of this exceptional sphere is a piece of the fibration and the knots are the intersection of the fibres of the almost contact pencil with it.

The local model described above allows us to prove the following

Lemma 6.5. *Let $z \in \mathbb{CP}^1$ be a coordinate, $(\mathbb{S}^3 \times \mathbb{D}^2, \xi_v(z))$ a $\mathbb{CP}^1$–family of contact structures on $\mathbb{S}^3 \times \mathbb{D}^2$ and $f_U : \mathbb{S}^3 \times \mathbb{D}^2 \longrightarrow \mathbb{CP}^1$ the map described above. Consider two open disks $\mathcal{B}_0, \mathcal{B}_\infty \subset \mathbb{CP}^1$, containing* 0 *and ∞ respectively such that the intersection $\mathcal{B}_0 \cap \mathcal{B}_\infty$ is an open annulus and the complement of $\mathcal{B}_0 \cap \mathcal{B}_\infty$ consists of two disjoint disks.*

There exists a homotopy $\xi_v^s(z)$ of $\mathbb{CP}^1$–families of plane fields, $s \in [0,1]$, such that

- *$\xi_v^0(z) = \xi_v(z)$, $\forall z \in \mathbb{CP}^1$.*
- *Near the boundary of $f_U^{-1}(z) \cong \mathbb{S}^1 \times \mathbb{D}^2$ and $\forall (z,s) \in \mathbb{CP}^1 \times [0,1]$, $\xi_v^s(z) = \xi_v(z)$.*
- *For any $z \in \mathbb{CP}^1$, the distribution $\xi_v^1(z)$ is an overtwisted contact structure on $f_U^{-1}(z)$ containing two disjoint Lutz tubes L_z^0 and L_z^∞ away from $\mathbb{S}^3 \times \{0\}$.*
- *There exist a smooth family of embedded overtwisted 2–disks $\mathcal{D}_z^0$ in L_z^0 for $z \in \mathcal{B}_0$ and $\mathcal{D}_z^\infty$ in L_z^∞ for $z \in \mathcal{B}_\infty$.*

Both $\mathcal{B}_0 \backslash \partial\mathcal{B}_0, \mathcal{B}_\infty \backslash \partial\mathcal{B}_\infty$ can be thought as neighborhoods of the upper and lower semi–spheres.

Proof. Let $h : \mathbb{S}^3 \longrightarrow \mathbb{CP}^1$ be the Hopf fibration, extend the fibration to $h : \mathbb{S}^3 \times \mathbb{D}^2 \longrightarrow \mathbb{CP}^1$ by projection onto the first factor. The idea is to use the exceptional divisor to create a couple of sections along $\mathcal{B}_0$ and $\mathcal{B}_\infty$. On the one hand, the exceptional divisor has a contact structure and we would rather not perturb around a small neighborhood of it. On the other hand the exceptional divisor is not $\mathbb{CP}^1$ but $\mathbb{S}^3$. Hence a global section cannot exist. We use two copies of the exceptional divisor away from $\mathbb{S}^3 \times \{0\} \subset \mathbb{S}^3 \times \mathbb{D}^2$ and we cover the base $\mathbb{CP}^1$ with the two disks $\mathcal{B}_0$, $\mathcal{B}_\infty$.

Let $q_0 = (1/2, 0)$, $q_\infty = (0, 1/2) \in \mathbb{D}^2$ be two fixed points and consider the two 3–spheres

$$\mathbb{S}_0^3 = \mathbb{S}^3 \times \{q_0\}, \qquad \mathbb{S}_\infty^3 = \mathbb{S}^3 \times \{q_\infty\}.$$

The fibre of the restriction of the fibration $(\mathbb{S}^3 \times \mathbb{D}^2, \xi_v(z)) \longrightarrow \mathbb{CP}^1$ to the submanifold $\mathbb{S}_0^3$ (resp. $\mathbb{S}_\infty^3$) is a transverse knot K_0^z (resp. K_∞^z). We will now insert two families of overtwisted disks.

Apply a full Lutz twist in a small neighborhood of each of those knots $K_0^z \in h^{-1}(z)$ parametrically on $z \in \mathbb{CP}^1$. This produces a 3–dimensional full Lutz twist on each fibre, see [**92, 61**]. This yields an $\mathbb{S}_0^3$–family of overtwisted disks parametrized as $\{\mathcal{D}_t^0\}_{t\in\mathbb{S}^3_0}$, thus we obtain a $\mathbb{S}^1$–family of overtwisted disks at each fibre. Note that the dependency of this parametric family of full Lutz twists on the point $z \in \mathbb{CP}^1$ is well–behaved. Indeed, let $i_z : K_z^0 \longrightarrow \mathbb{S}^3_0$ be the injection and consider coordinates (ρ, φ) in the normal bundle of this embedding. In a small neighborhood of the zero section, the contact structure reads

$$\xi_v(z) = \ker\{i_z^*\alpha_{\mathbb{S}^3} + \rho^2 d\varphi\}.$$

The pair of functions (h_1, h_2) used in Section 4.3 [**61**] to perform the full Lutz twist can be made ρ–dependent. Thus the resulting contact structure has the form

$$\xi_v^1(z) = \ker\{h_1(\rho) \cdot i_z^*\alpha_{\mathbb{S}^3} + h_2(\rho) \cdot \rho^2 d\varphi\}.$$

This clarifies the dependency of the construction with respect to $z \in \mathbb{CP}^1$.

Perform the same twist procedure for the family of knots $K_\infty^z \in h^{-1}(z)$ to obtain another family of overtwisted disks $\{\mathcal{D}_t^\infty\}_{t\in\mathbb{S}^3_\infty}$. The two families of disks can indeed be assumed disjoint by letting the radius in which we perform the full Lutz twists be small enough. The support of the pair of full Lutz twists can be chosen not to intersect the exceptional divisor and be contained in the interior of $\mathbb{S}^3 \times \mathbb{D}^2$. This construction provides the homotopy in the statement of the Lemma. See Figure 4.

We need the base $\mathbb{CP}^1$ to be the parameter space instead of the 3–spheres $\mathbb{S}^3_0$ and $\mathbb{S}^3_\infty$. Restricted to $\mathcal{B}_0$ or $\mathcal{B}_\infty$ the Hopf fibration becomes trivial and therefore there exist two sections $s_0 : \mathcal{B}_0 \longrightarrow \mathbb{S}^3 \cong \mathbb{S}^3_0$ and $s_\infty : \mathcal{B}_\infty \longrightarrow \mathbb{S}^3 \cong \mathbb{S}^3_\infty$. The required families are defined as

$$\{\mathcal{D}_z^0\} = \{\mathcal{D}^0_{s_0(z)}\}, z \in \mathcal{B}_0,$$

$$\{\mathcal{D}_z^\infty\} = \{\mathcal{D}^\infty_{s_\infty(z)}\}, z \in \mathcal{B}_\infty.$$

Note that the two families of overtwisted disks are disjoint since the two families of Lutz twists are. Further, there exists a small neighborhood of the exceptional divisor $\mathbb{S}^3 \times \{0\}$ where no deformation is performed. The statement of the Lemma follows. □

The global construction can be simply achieved:

Proof of Proposition 6.4. Apply Lemma 6.5 to a neighborhood of one exceptional sphere $E_0 \in E = \{E_0, E_1, \dots, E_s\}$. The families of overtwisted

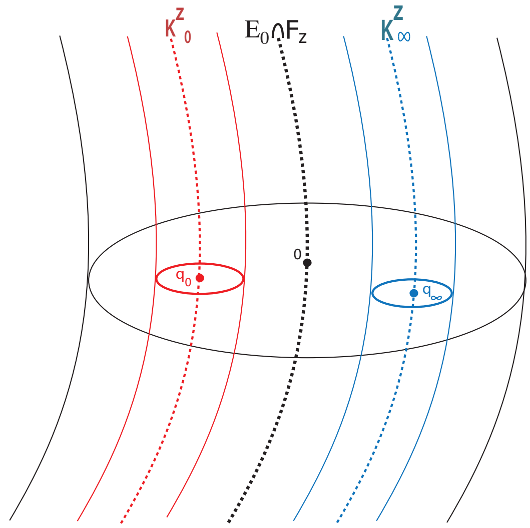


FIGURE 4. The neighborhood of the exceptional divisor intersected with a fibre F_z. The cylinder on the left (with axis K_0^z) is the support of the full Lutz twist around the knot $K_0^z \cong \mathbb{S}^1 \times \{q_0\}$ and the cylinder on the right (with axis K_∞^z) corresponds to the support of the full Lutz twist around the knot $K_\infty^z \cong \mathbb{S}^1 \times \{q_\infty\}$.

disks do not meet C or any E_j. Indeed, the two families are arbitrarily close to E_0 and the exceptional divisors are pairwise disjoint and none of them intersect the critical curves C. Thus, maybe after shrinking the neighborhood $U(E_0)$ in the construction, the families are located away from $U(f)$. $\square$

Thus we obtain the families of overtwisted disks required to apply Theorem 6.2. The vertical deformation is described using a suitable cell decomposition of the base $\mathbb{CP}^1$. The vertical contact condition is ensured progressively above the 0–cells, the 1–cells and the 2–cells.

6.3. Adapted families. Let (f, C) be an almost contact fibration. A finite set of oriented immersed connected curves T in $\mathbb{CP}^1$ will be called an adapted family for (f, C) if it satisfies the following properties:

- The image of the set of critical values $f(C)$ is part of T.
- Given any element $c \in T$, there exists another element of $c' \in T$ having a non–empty intersection[1] with c. Any two elements of T intersect transversally.
- There exists no triple intersection point between the curves of T.

[1]In case c has a self–intersection, then $c' = c$ is allowed.

- The complement $\mathbb{CP}^1 \setminus |T|$ is a union of open disks.

$|T| \subset \mathbb{CP}^1$ denotes the underlying set of points of the elements of T. The elements of an adapted family T that are not in the image of a component of C are referred to as fake components. Let $N \in \mathbb{N}$ be fixed. The insertion of fake curves proves the existence of an adapted family with $\mathrm{diam}_{g_0}(\mathbb{CP}^1 \setminus |T|) \leq 1/N$, g_0 the standard round metric.

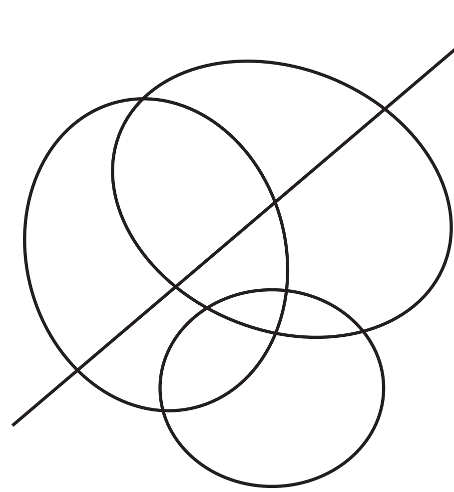


FIGURE 5. Part of an adapted family T. The associated subdivision consists of certain 2–cells with their boundaries being a union of parts of various elements in the family T.

There is a cell decomposition of $\mathbb{CP}^1$ associated to an adapted family, the 1–skeleton being $|T|$. See Figure 5. In order to conclude Theorem 6.1 we shall first deform in a neighborhood of each vertex relative to the boundary, proceed with a neighborhood of the 1–cells and finally obtain the vertical contact condition in the 2–cells. To be precise in the description of the procedure, we introduce some notation. This is not strictly necessary but it provides the adequate pieces in the framework to apply Eliashberg's result.

Let $L_j \in T$ be a curve, $U(L_j)$ be an open tubular neighborhood and denote

$$\partial\overline{U(L_j)} = L_j^0 \cup L_j^1.$$

Suppose that $\bigcup_{j\in J}|L^i_j|$ is isotopic to $|T|$ for both $i=0,1$; this can be achieved by taking a small enough neighborhood of each L_j. See Figure 6. We use $V(L_j)$ to denote a slightly larger tubular neighborhood satisfying this same condition. Fix an intersection point p of two elements $L_j, L_k \in T$. Denote by $\mathcal{A}_p$ the connected component of the intersection of $U(L_j)\cap U(L_k)$ containing p. Similarly, let $\mathcal{VA}_p$ be the connected component of the intersection of $V(L_j)\cap V(L_k)$ that contains p, and denote $\mathcal{AA}_p = \mathcal{VA}_p\backslash\mathcal{A}_p$.

Consider a small neighborhood $U(T)$ of $|T|$. The open connected components of

$$U(T)\backslash\{\cup\mathcal{A}_p\}$$

are homeomorphic to rectangles $\mathcal{B}_i$, p being treated as an index over the intersection points. A suitable indexing for i is also assumed. The third class of pieces constitute the interior of the complement in $\mathbb{CP}^1$ of the open set formed by the union of the sets $\mathcal{A}_p$ and $\mathcal{B}_i$. Its connected components are denoted $\mathcal{C}_l$. Thus, neighborhoods of the 0–cells, 1–cells and 2–cells are labeled $\mathcal{A}_p$, $\mathcal{B}_i$ and $\mathcal{C}_l$ respectively. See Figure 6.

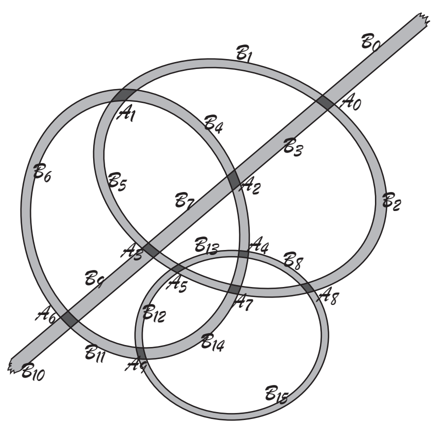


FIGURE 6. The sets A_p and B_i associated to the subdivision of the figure 5. The sets A_p are drawn in darker grey.

Finally, we define the sets $\mathcal{BB}_i$. Let $\mathcal{B}_i$ connect a couple of open sets[2] of the form $\mathcal{A}_p$. There exists a curve $L_{\mathcal{B}_i}$ contained in $\mathcal{B}_i$ which is a part of a curve $L_i \in T$. $L_{\mathcal{B}_i}$ is part of a 1–cell in the decomposition associated to

[2] Both sets may be the same for the self–intersecting curves.

the adapted family T. Let $L^0_{\mathcal{B}_i}$ and $L^1_{\mathcal{B}_i}$ denote the two boundary components of $\overline{\mathcal{B}}_i$ which are part of the curves L^0_i and L^1_i defined above. Then we declare $\mathcal{BB}^0_i$ (resp. $\mathcal{BB}^1_i$) to be the connected component of $V(L_i)\backslash\mathcal{B}_j$ containing the boundary curve L^0_i (resp. L^1_i). Their union $\mathcal{BB}^0_i \cup \mathcal{BB}^1_i$ will be denoted $\mathcal{BB}_i$. See Figures 7 and 8.

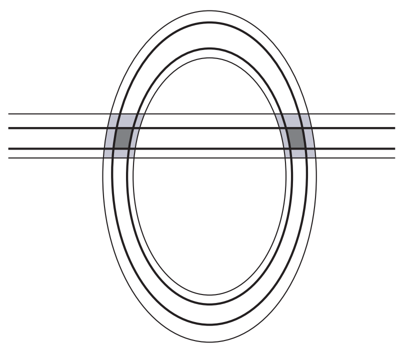


FIGURE 7. Example of two components $\mathcal{VA}_p$ and $\mathcal{VA}_q$ in light gray, containing $\mathcal{A}_p$ and $\mathcal{A}_q$, in dark gray.

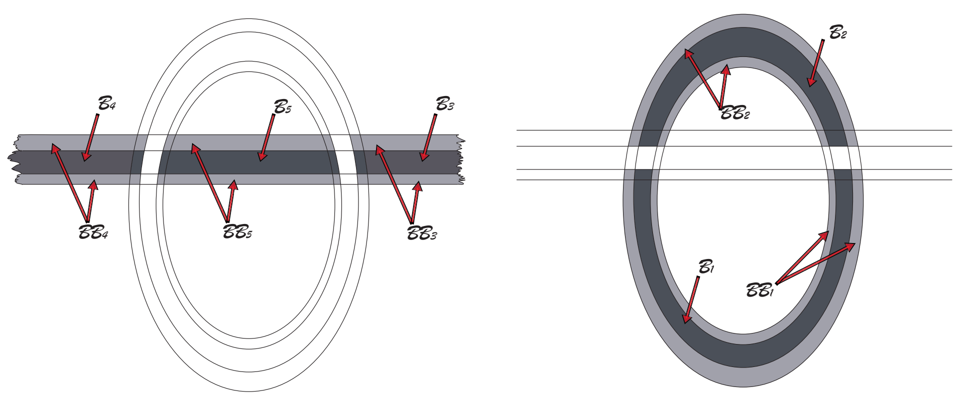


FIGURE 8. Example of the sets $\mathcal{B}_i$ and $\mathcal{BB}_i$ for the subdivision of Figure 7.

6.4. The vertical construction. In this subsection we prove Theorem 6.1. The following lemma is a simple exercise in differential topology and can be considered as a particular case of Ehresmann's fibration theorem. It will be used in the proof of Theorem 6.1. We include it for completeness.

LEMMA 6.6. *Let $f : E \longrightarrow \mathbb{D}^2$ be a locally trivial smooth fibration over the unit disk with compact fibres E_z, $z \in \mathbb{D}^2$. Decompose ∂E along its corners as $\partial E = f^{-1}(\partial \mathbb{D}^2) \cup \partial_h E$ and suppose that $\partial_h E$ is a smooth closed boundary. Suppose also that there is a collar neighborhood N of $\partial_h E$ and a closed submanifold S such that restricting f to S and N induces locally trivial fibrations. Let S_0, N_0 be their fibres over $0 \in \mathbb{D}^2$.*

Then there exists a diffeomorphism $g : E \longrightarrow E_0 \times \mathbb{D}^2$ making the following diagram commute

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d, "\pi"'] & E_0 \times \mathbb{D}^2 \arrow[d, "\pi_0"] \\ \mathbb{D}^2 \arrow[r, equal] & \mathbb{D}^2 \end{tikzcd}$$

such that $g(N) = N_0 \times \mathbb{D}^2$ and $g(S) = S_0 \times \mathbb{D}^2$.

PROOF. Let g be Riemannian metric in E such that $(TE_z)^{\perp g} \subset TS$ and $(TE_z)^{\perp g} \subset T(\partial_h E)$, for the points z where the condition can be satisfied. Let $X = \partial_r$ be the radial vector field in $\mathbb{D}^2 \backslash \{0\}$ and construct the connection H_π associated to the Riemannian fibration:

$$H_\pi(e) = (T_e F_{\pi(e)})^{\perp g}.$$

The condition imposed on the Riemannian metric implies that $\partial_h E$ and S are tangent to the horizontal connection H_π. Let $\widetilde{X}$ be a lift of X through H_π and $\phi_t(e)$ the flow of this vector field. Define

$$\begin{array}{rcl} E & \xrightarrow{g} & E_0 \times \mathbb{D}^2 \\ e & \longmapsto & (\phi_{(-||\pi(e)||)}(e), \pi(e)). \end{array}$$

This map satisfies the required properties. $\square$

Proof of Theorem 6.1. Let (f, C, E) be a good ace fibration and T an adapted family to (f, C, E). Note that a horizontal complement H is defined away from $U(C)$ and provides the splitting specified in (6.2). Proposition 6.4 and choose $\mathcal{B}_0$ and $\mathcal{B}_\infty$ in the statement such that $\partial \mathcal{B}_0$ and $\partial \mathcal{B}_\infty$ are both contained in two different 2–cells $\mathcal{C}_0$ and $\mathcal{C}_\infty$. Lemma 2.5 implies that this procedure preserves the homotopy class of (M, ξ, ω).

In order to establish Theorem 6.1 we need to perform a deformation which is fixed in a neighborhood of $U(C)$ and leaves the distribution H unchanged, i.e. it should be a strictly vertical deformation.

Deformation at the 0*–cells*: Let p be a vertex with neighborhood $\mathcal{A}_p$ and

$$\mathcal{F} = f^{-1}(\mathcal{VA}_p) \setminus (f^{-1}(\mathcal{VA}_p) \cap U(C)).$$

We can assume that $\mathcal{VA}_p$ is small enough and choose a neighborhood $U(C)$ such that the map f restricts to a trivial fibration on $\mathcal{F}$ and induces a fibration on $\partial\mathcal{F}$. Consider a trivialization of the former fibration over $\mathcal{VA}_p$. The manifolds with boundary $\mathcal{F}_z = f^{-1}(z)\setminus(f^{-1}(z) \cap U(C))$ are all diffeomorphic. Let N_z be a collar neighborhood of $\partial\mathcal{F}_z$ in which the distribution is contact. Given an exceptional divisor $E_i \in E$ denote by $U(E_i)_z$ the intersection of $U(E_i)$ with the fibre $\mathcal{F}_z$. Applying the trivializing diffeomorphism provided in Lemma 6.6, we may assume $\mathcal{F}_z \times \mathcal{VA}_p \cong \mathcal{F}$, $U(E_i)_z \times \mathcal{VA}_p \cong U(E_i)$ and $N_z \times \mathcal{VA}_p \cong N$.

Thus we have a manifold with boundary $\mathcal{F}$ with a family of distributions ξ_z parametrized by the topological disk $\mathcal{VA}_p$ containing $K = \mathcal{A}_p$. Also a good set G of submanifolds that are already contact for any contact fibre over $\mathcal{VA}_p$. The good set G consists of the union of N, $U(E_j)$ and a neighborhood of one of the two overtwisted disks[3]. Let us say $p \in \mathcal{B}_0$ and we choose a neighborhood of $\mathcal{D}^\infty$. A neighborhood of this set will not be perturbed. The remaining disk $\mathcal{D}^0$ is contactomorphic to the standard overtwisted disk for each element of the family of distributions. This set–up satisfies the hypotheses of Corollary 6.2. It should be applied to a smaller parameter space K and then Corollary 6.3 is used with $\lambda = \emptyset$ to obtain a deformation relative to the boundary. Since we are able to obtain a deformation relative to the boundary we may perform the deformation at each neighborhood of the 0–cells and extend trivially to the complement of $\mathcal{VA}_p$ in $\mathbb{CP}^1$.

Deformation at the 1*–cells*: Almost the same strategy applied to the 0–cells applies, although we should not undo the deformation in a neighborhood of the 0–cells. Corollaries 6.2 and 6.3 allow us to perform deformations relative to a subfamily, so in this case λ will be non–empty. See Figure 9.

Deformation at the 2*–cells*: In this situation Theorem 6.2 also applies after a suitable trivialization of the smooth fibration provided by Lemma 6.6. Note that in this case the fibres do not have the boundary contribution of $U(C)$ since its image is not contained in the 2–cells. The set L

[3]These disks are trivialized along with N using Lemma 6.6.

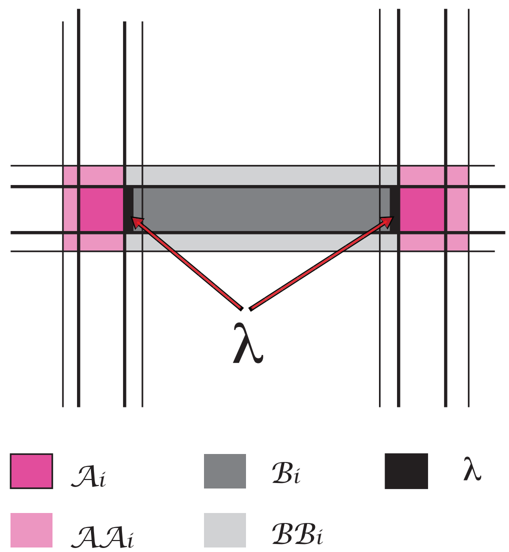


FIGURE 9. The distributions set $\xi_z \subset \mathcal{BB}_i$ with $z \in \lambda$ are already contact distributions.

is a small tubular neighborhood of the boundary of the 2–cells. Except at $\mathcal{C}_0$ and $\mathcal{C}_\infty$, we may use any of the two families of overtwisted disks to apply the result. Let it be $\mathcal{D}_z^0$. In the remaining family the distributions are contact and so we include the disks in the set G, that also contains N and $U(E_i)$. At $\mathcal{C}_0$ we use the family $\mathcal{D}_z^0$, since it is the only one well–defined over the whole set. Proceed analogously at $\mathcal{C}_\infty$. Note that this argument is possible because the deformation is relative to the boundary. Then Theorem 6.2 applies to the 2–cells and we extend trivially the deformation. We obtain a vertical contact distribution $(F_z, \widetilde{\xi}_z)$ away from $U(C)$.

In order to conclude the statement of the Theorem, consider the direct sum $\widetilde{\xi}_z \oplus H$ to include the critical set, which has not been deformed. This is the required vertical contact structure. Notice that this construction preserves the almost contact class of the distribution since it is performed homotopically only in the vertical direction. Hence Lemma 2.5 provides a homotopy on the complement of $U(C)$ relative to the boundary. This yields a homotopy over the manifold M. $\square$

7. Horizontal Deformation I

Consider an almost contact distribution (M, ξ, ω) and a good ace fibration (f, C, E) with associated adapted family T. Theorem 6.1 deforms ξ to a vertical contact structure with respect to (f, C, E). To obtain a honest contact structure the distribution has to be suitably changed in the horizontal direction. As in the previous section, this is achieved in three stages. The content of this Section consists of the first two of these: deformation in the pre–image of a neighborhood of the 0– and the 1–cells of the adapted family T. The main result of this Section is the following theorem.

THEOREM 7.1. *Let (M, ξ, ω) be a vertical contact structure with respect to a good ace fibration (f, C, E) and T an adapted family. Then there exists a homotopic deformation (ξ', ω') of (ξ, ω) relative to C and E such that (f, C, E) is a good ace fibration for (ξ', ω'), (ξ', ω') is a vertical contact almost contact structure and ξ' is a contact structure in the pre–image of a neighborhood of $|T|$.*

The vertical distribution is fixed along the deformation. In this sense the deformation in the statement is horizontal. The fibration (f, C, E) will not be deformed to prove this fact, just the almost contact structure.

Theorem 7.1 follows Proposition 7.6 and Lemma 2.5. To prove the statement we trivialize the vertical contact fibration over a neighborhood of the 0–cells. Then the deformation is performed using an explicit local model. The deformation in a neighborhood of the 0–cells is the content of Proposition 7.5. Then we proceed with the pre–image of a neighborhood of the 1–cells. This is Proposition 7.6. The same local model is used in both deformations.

7.1. Local model. The following lemma is used to prove Proposition 7.5 and Proposition 7.6. It is a version of results in Section 2.3 of [**45**] concerning deformations of a family of distributions near the 1 and 2–skeleta of a 3–manifold. The connectedness condition is stated there as the vanishing of a relative fundamental group.

LEMMA 7.1. *Let (F, ξ_t) be a family of contact structures over a compact 3–manifold F parametrized by $(s, t) \in [-\varepsilon, \varepsilon] \times [0, 1]$ with ξ_t is constant along the s–lines and α_t associated contact forms. Consider the projection*

$$F \times [-\varepsilon, \varepsilon] \times [0, 1] \xrightarrow{\pi} F \times [0, 1],$$

and the distribution ξ on $F\times[-\varepsilon,\varepsilon]\times[0,1]$ defined globally by the kernel of the form

$$\alpha_H(p,s,t)=\alpha_t+H(p,s,t)dt, \qquad H\in C^\infty(F\times[-\varepsilon,\varepsilon]\times[0,1]).$$

Suppose that $|H(p,s,t)|\leq c\cdot|s|$ and assume that the 1–form α_H is a contact form in a compact set G such that the intersection of G with any segment $\{p\}\times[-\varepsilon,\varepsilon]\times\{t\}$ is either connected or empty.

Then, there is a small perturbation $\widetilde{H}$ of H relative to G such that $\alpha_{\widetilde{H}}$ defines a contact structure. In precise terms, $|\widetilde{H}-H|\leq 3c\varepsilon$ and $\widetilde{H}|_G=H|_G$.

Proof. Let us compute the contact condition on $\alpha=\alpha_H$.

$$d\alpha=d\alpha_t+dt\wedge\partial_t\alpha_t+dH\wedge dt \quad\Longrightarrow\quad (d\alpha)^n=(d\alpha_t)^n+(d\alpha_t)^{n-1}\wedge dH\wedge dt.$$

Therefore, the contact condition is described as

$$(d\alpha)^n\wedge\alpha=(d\alpha_t)^{n-1}\wedge\alpha_t\wedge(\partial_sH\cdot ds\wedge dt).$$

Thus, the 1–form α is a contact form if and only if $\partial_sH>0$.

Given $(p,t)\in F\times[0,1]$, $\pi^{-1}(p,t)$ is a 4–parametric family of 1–dimensional manifolds. The connectedness of $\pi^{-1}(p,t)\cap G$ and the compactness of G assure that it is possible to perturb H to an $\widetilde{H}$ relative to G and satisfying the contact condition. Indeed, the connectedness condition allows us to perturb the function H on at least one end of the curves in $F\times[-\varepsilon,\varepsilon]\times[0,1]$ and obtain a function $\widetilde{H}$ with $\partial_s\widetilde{H}>0$. □

7.2. Contact connections. The previous Lemma 7.1 can be used if the contact form has the expression as in the hypotheses of the statement. This is achieved with the choice of an appropriate trivialization obtained by parallel transport. It is convenient to review the notions introduced in [**88**].

Definition 7.2. A contact fibration is a smooth fibration $\pi:M\longrightarrow B$ with a co–oriented codimension–1 distribution $\xi\subset TM$ such that the intersection of ξ with any fibre induces a contact structure on that fibre.

Consider a contact fibration (π,ξ), a 1–form α such that $\xi=\ker\alpha$ and the vertical bundle $\ker\pi$. A contact fibration has an associated contact connection H_ξ. It is defined as the orthogonal of the symplectic subbundle $(\ker\pi\cap\xi, d\alpha|_{\ker\pi\cap\xi})$ in ξ with respect to $d\alpha|_\xi$. Note that the contact connection only depends on the contact structure and not on the choice of the contact form.

LEMMA 7.3. *Let (π, ξ) be a contact fibration. The parallel transport with respect to a contact connection is by contactomorphisms.*

This is a simple computation. See [**88**], [**113**]. A vertical contact almost contact structure (M, ξ, ω) with respect to a good ace fibration (f, C, E) is in particular a contact fibration away from the critical locus C. Suppose that $\xi = \ker \alpha$ and let $\xi_v = \ker \alpha_v$ be the vertical distribution. The symplectic structure ω and $d\alpha|_\xi$ both provide a horizontal complement for the vertical distribution ξ_v in ξ. These are defined as the annihilators of the vertical bundles with respect to the 2–forms ω and $d\alpha|_\xi$. Let us denote the first one by H_ω and note that the second one is the contact connection H_ξ introduced above. The distribution H_ξ is not necessarily symplectic for ω. Consider a symplectic structure ω_ξ for H_ξ coinciding with the symplectic structure $d\alpha|_{H_\xi}$ on a neighborhood of C and E. Then $(M, \xi, d\alpha_v \oplus \omega_\xi)$ is a vertical contact almost contact structure for (f, C, E). Lemma 2.5 implies the following

LEMMA 7.4. *Let (M, ξ, ω) be a vertical contact almost contact structure with respect to a good ace fibration (f, C, E), α_v such that $\xi_v = \ker \alpha_v$ and ω_ξ a symplectic structure for the contact connection associated to (f, ξ). Then (M, ξ, ω) and $(M, \xi, d\alpha_v \oplus \omega_\xi)$ are homotopic almost contact structures.*

In order to be able to apply Lemma 7.1 we need a deformation of (M, ξ, ω) such that at least in one direction the parallel transport along the deformed almost contact connection is a contactomorphism. This allows us to trivialize with the almost contact connection and obtain a vertical contact distribution constant along that direction. Thus conforming the hypotheses of Lemma 7.1. Both Lemmas 7.3 and 7.4 provide such a construction. The following two subsections provide details.

7.3. Deformation along intersection points. In this subsection we obtain a contact structure in a neighborhood of the fibres over a neighborhood of the intersection points of an adapted family T. The precise statement reads as follows:

PROPOSITION 7.5. *Let (M, ξ, ω) be a vertical contact structure with respect to a good ace fibration (f, C, E) and T an adapted family. Then there exists a deformation (ξ', ω') of (ξ, ω) relative to C and E such that*

(f, C, E) is a good ace fibration for (ξ', ω') and ξ' is a contact structure in the pre–image of a neighborhood of the 0–cells of $|T|$.

PROOF. Let z be a point of intersection of the adapted family T, (ϕ, U) a sufficiently small chart centered at z with the diffeomorphism $\phi : U \longrightarrow [-1,1] \times [-1,1]$, Cartesian coordinates $(s,t) \in [-1,1] \times [-1,1]$ and $N = f^{-1}(U) \backslash U(f)$. The geometric argument to prove the statement is simple. Lemmas 7.4 and 7.3 are used to trivialize f over a neighborhood of the 0–cells such that the hypotheses of Lemma 7.1 can be applied. Let us provide the details.

The map $f : N \longrightarrow U$ is a smooth trivial fibration with fibre F. Lemma 6.8 provides an adequate trivializing diffeomorphism $g : N \longrightarrow F \times [-1,1] \times [-1,1]$. Let $(\lambda, \Omega) = (g_*\xi, g_*\omega)$ be the almost contact structure in this local model and

$$f_\lambda = \phi \circ f \circ g^{-1} : F \times [-1,1] \times [-1,1] \longrightarrow [-1,1] \times [-1,1],$$
$$\text{defined by } f_\lambda(p) = (\sigma(p), \tau(p)).$$

This is a contact fibration for the distribution λ and the almost contact structure (λ, Ω) is a contact structure near $g(\partial N \setminus f^{-1}(\partial U))$. Consider the 1–forms α and α_v defining the distributions λ and λ_v. Lemma 7.4 allows us to deform the symplectic structure Ω to $d\alpha_v \oplus \Omega_\lambda$ for a suitable choice of symplectic structure Ω_λ in the $d\alpha$–orthogonal of λ_v in λ. Lemma 7.3 implies that the parallel transport along the lift of the vector field ∂_s to the connection H_λ consists of contactomorphisms. This provides a specific trivialization such that the contact form satisfies the hypotheses of Lemma 7.1.

Indeed, consider the connection H_λ for the fibration f_λ and the vector field ∂_s in the base $[-1,1] \times [-1,1]$. Let X_s be the lift of ∂_s to H_λ and m_p^τ the parallel transport along the segment

$$\gamma : [0, \tau] \longrightarrow [-1,1] \times [-1,1], \quad \gamma(r) = p + (r, 0).$$

That is, m_p^τ is the time–τ flow of X_s. There exists a small $\varepsilon \in \mathbb{R}^+$ such that the flow m_p^τ is well–defined for all $|\tau| < \varepsilon$ and $p \in \{0\} \times [-1,1]$. This might require a perturbation of the trivializing diffeomorphism g along a neighborhood of the boundary $f_\lambda^{-1}(\partial((-\varepsilon, \varepsilon) \times [-1,1]))$.

In order to obtain the required trivialization consider the diffeomorphism

$$\iota : F \times (-\varepsilon, \varepsilon) \times [-1,1] \longrightarrow F \times (-\varepsilon, \varepsilon) \times [-1,1],$$

$$\text{defined by } p \longmapsto \iota(p) = (m_{(0,\tau(p))}^{-\sigma(p)}(p), f_\lambda(p)).$$

The lift of the direction ∂_s is part of the trivialized distribution. In precise terms, the push–forward of ξ in $g^{-1}(F \times (-\varepsilon, \varepsilon) \times [-1,1])$ along $\iota \circ g$ is a distribution $(\iota \circ g)_* \xi$ given by the kernel of a 1–form

$$\alpha_{(s,t)} + H(p,s,t)dt, \text{ satisfying } \partial_s \alpha_{(s,t)} = 0.$$

Lemma 7.1 can then be applied. The good set G is chosen to be a suitable neighborhood of the trivialization of the boundary $\partial F \times (-\varepsilon, \varepsilon) \times [-1,1]$. The statement of the Lemma yields a smooth function

$$\widetilde{H} : F \times (-\varepsilon, \varepsilon) \times [-1,1] \longrightarrow \mathbb{R}$$

inducing a contact structure in this local model.

The previous procedure has to be considered inside the manifold. We should then perform the perturbation relative to the boundary of the base $(-\varepsilon, \varepsilon) \times [-1,1]$. To this aim, consider $\delta \in \mathbb{R}^+$ small enough and a smooth cut–off function $c_\delta : [-1,1] \longrightarrow [0,1]$ satisfying

$$c_\delta(x) = 1 \text{ for } |x| \leq \delta, \quad c_\delta(x) = 0 \text{ for } |x| \geq 1 - \delta.$$

Then the interpolating function

$$h(p,s,t) = c_\delta(\varepsilon^{-1}s)c_\delta(t)\widetilde{H}(p,s,t) + (1 - c_\delta(\varepsilon^{-1}s)c_\delta(t))H(p,s,t)$$

induces the form $\alpha = \alpha_{(s,t)} + h(p,s,t)dt$ which coincides with $\alpha_{(s,t)} + H(p,s,t)dt$ near the boundary of $(-\varepsilon, \varepsilon) \times [-1,1]$. The perturbation can thus be made relative to the boundary and inserted in the manifold. The deformation from the initial distribution to that defined by the contact form α satisfies the statement of the Proposition. □

7.4. Deformation along curves. Once we have achieved the contact condition in a neighborhood of the fibres over the 0–skeleton, we proceed with a neighborhood of the fibres over the 1–skeleton.

PROPOSITION 7.6. *Let (M, ξ, ω) be a vertical contact structure with respect to a good ace fibration (f, C, E), T an adapted family and $\mathbb{T}$ a neighborhood of T. Suppose that (M, ξ) is a contact structure on a neighborhood $\mathbb{O}$ of the fibres over the 0–cells of T. Then there exists a deformation (ξ', ω') of (ξ, ω) relative to C, E and $\mathbb{O}$ such that (f, C, E) is a good ace fibration for (ξ', ω') and ξ' is a contact structure in the pre–image of $\mathbb{T}$.*

Let $\mathbb{S}$ be a small neighborhood of the set of fibres over $\mathbb{T}\backslash\mathbb{O}$. See Figure 10. The argument applied over $\mathbb{O}$ in the previous subsection works

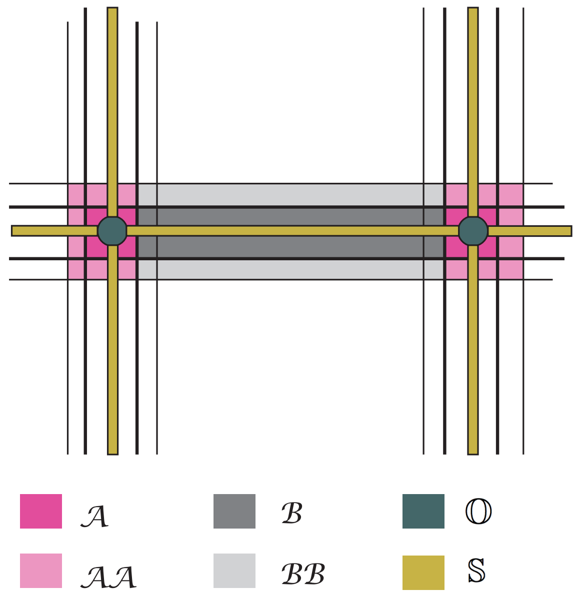


FIGURE 10. The deformation domains.

analogously when applied to $\mathbb{S}$. Thus, no detailed proof is given. The only subtlety lies in the appropriate choice of the compact set G when Lemma 7.1 is applied.

Let $z, w \in \mathbb{CP}^1$ with corresponding neighborhood $\mathbb{O}_z, \mathbb{O}_w$; we focus on a line segment $S \subset |T|$ joining these two points. Let (ϕ, U) be a local chart around $S\backslash(\mathbb{O}_z \cup \mathbb{O}_w)$ with cartesian coordinates (s, t) such that

$$\phi(U) = [-\varepsilon, \varepsilon] \times [0,1], \qquad \phi(S) = \{0\} \times [0,1].$$

LEMMA 7.7. *There exist an arbitrarily small neighborhood $\mathbb{S}$ of S and a horizontal deformation of the vertical contact almost contact structure (ξ, ω) supported in the pre–image of $\mathbb{S}$, relative to the pre–images of $\mathbb{S} \cap \mathbb{O}_z$ and $\mathbb{S} \cap \mathbb{O}_w$, and conforming the following properties:*

- *The deformation is relative to $U(f)$ where ξ is a contact structure.*
- *There exists a local chart (ϕ, U) such that the parallel transport of the associated almost contact connection along the vector field $\phi^*\partial_s$ consists of contactomorphisms.*

This follows from subsection 7.2.

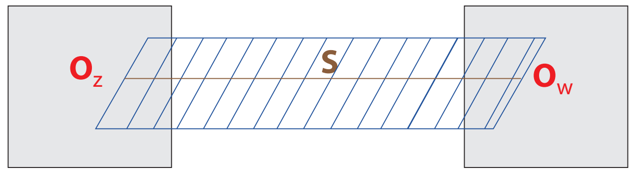


FIGURE 11. The deformation curves $\phi^*\partial_s$.

Proof of Proposition 7.6. Use Lemma 7.7 to ensure that the parallel transport along the lift of ∂_s is by contactomorphisms. Choose the s–coordinate in the neighborhood $\mathbb{S}$ in such a way that the curves which provide the lift of $\phi^*\partial_s$ either have at most one of the ends in the fibres over a small neighborhood of the 0–skeleton or are contained therein. See Figure 11. This allows us to choose a compact set G containing the fibres over the two endpoints plus a neighborhood of the boundary of all the fibres such that the intersection of G with any such arc is connected. There might be the need to progressively shrink the neighborhoods of the fibres over the 0–skeleton. Apply Lemma 7.1 to produce a contact structure in a neighborhood of the fibres over the 1–skeleton without perturbing the existing contact structure in a small neighborhood of fibres over the endpoints. $\square$

8. Fibrations over the 2–disk.

Let (F, ξ_v) be a contact 3–manifold, $\xi_v = \ker \alpha_v$ and $\mathbb{D}^2$ a 2–disk. In this Section we study contact structures on the product manifold $F \times \mathbb{D}^2$. Consider the coordinates $(p, r, \theta) \in F \times \mathbb{D}^2$. The previous sections essentially reduce Theorem 1.1 to the existence of a contact structure on $F \times \mathbb{D}^2$ restricting to a prescribed contact structure on a neighborhood of the boundary $F \times \partial\mathbb{D}^2$. See Theorem 9.1 in Section 9 for details on the end of the proof.

Fix an $\varepsilon \in (0,1)$ and consider $H \in C^\infty(F \times \mathbb{D}^2(1))$ to be a smooth function such that $\partial_r H > 0$ for $r \in (1-\varepsilon, 1]$. Then the 1–form

$$\alpha = \alpha_v + H(p,r,\theta)d\theta$$

defines a distribution $\xi = \ker\alpha$. It can be endowed with the symplectic form

$$\omega = d\alpha_v + (1-\tau(r)) \cdot rdr \wedge d\theta + \tau(r)dH \wedge d\theta,$$

where $\tau : [0,1] \longrightarrow [0,1]$ is an strictly increasing smooth function such that

$$\tau(x) = 0 \text{ for } x \in [0, 1-\varepsilon] \text{ and } \tau(x) = 1 \text{ for } x \in [1-\varepsilon/2, 1].$$

Then (ξ,ω) is an almost contact structure on $F \times \mathbb{D}^2(1)$ which is a contact structure on the neighborhood $F \times (1-\varepsilon/2, 1] \times \mathbb{S}^1$ of the boundary $F \times \partial\mathbb{D}^2(1)$.

The main result in this Section is the following:

THEOREM 8.1. *Let (F,ξ_v) be a contact 3–manifold with $c_1(\xi_v) = 0$, $\xi_v = \ker\alpha_v$ and L a transverse link. Given $\varepsilon \in (0,1)$, consider a function $H \in C^\infty(F \times \mathbb{D}^2(1))$ such that $\partial_r H > 0$ in $r \in (1-\varepsilon, 1]$ and $H|_{L\times\mathbb{D}^2(1)} \geq 0$, and the almost contact structure*

$$(\xi,\omega) = (\ker(\alpha_v + H(p,r,\theta)d\theta), d\alpha_v + (1-\tau(r)) \cdot rdr \wedge d\theta + \tau(r)dH \wedge d\theta),$$

where τ is the function described above.

Then there exists a 1–parametric family of almost contact structures $\{(\xi_t,\omega_t)\}$, constant along the boundary $F \times \partial\mathbb{D}^2(1)$ and with $(\xi_0,\omega_0) = (\xi,\omega)$ such that:

- *a. $(\xi_1,\omega_1) = (\ker\alpha, d\alpha)$ is a contact structure for some contact form α on $F \times \mathbb{D}^2(1)$.*
- *b. The submanifold $L \times \mathbb{D}^2(1)$ is a contact submanifold of $(F \times \mathbb{D}^2(1), \xi_1)$ and the induced contact structure is a small neighborhood of a full Lutz twist along $L \times \{0\}$.*

In coordinates $(z,r,\theta) \in L \times \mathbb{D}^2(1)$, the contact structure obtained by a full Lutz twist in a neighborhood $\mathcal{N}(L) \cong L \times \mathbb{D}^2$ of L along $L \times \{0\}$ is described as

$$\xi_{|L\times\mathbb{D}^2(1)} = \ker(\cos(2\pi r)dz + r\sin(2\pi r)d\theta).$$

Consider the domain $L \times \mathbb{D}^2(5/4)$ with the previous equation defining the contact structure. The term *small neighborhood* of a full Lutz twist refers to an open subset $U \cong L \times \mathbb{D}^2(1)$ such that it can be contact embedded as $L \times \mathbb{D}^2(1) \subset U \subset L \times \mathbb{D}^2(5/4)$.

This theorem is used to conclude Theorem 1.1 in Section 9. In brief, it is used to deform the almost contact structure over the 2–cells of the decomposition associated to an adapted family T of a vertical good ace fibration (f, C, E). In this description of the fibration over the 2–cells, the part corresponding to the exceptional divisors is the submanifold $L \times \mathbb{D}^2(1)$. Although the deformation in the statement is not relative to a neighborhood of them, the resulting contact structure is described in the part b. of Theorem 8.1.

Example. Suppose that the function $H \in C^\infty(F \times \mathbb{D}^2(1))$ also satisfies

$$H(p, 1, \theta) > 0, \text{ for all } (p, \theta) \in F \times \mathbb{S}^1.$$

The contact condition for the initial form $\alpha_v + H(p, r, \theta)d\theta$ is $\partial_r H > 0$. Consider a smooth family $\{H_t\}_{t\in[0,1]}$ of functions in $F \times \mathbb{D}^2(1)$ such that

$$H_0 = H, \quad H_1(p, 0, \theta) = 0, \quad \partial_r H_1 > 0 \text{ for } r \in (0, 1]$$

$$\text{and } H_t(p, 1, \theta) = H_0(p, 1, \theta).$$

Suppose that H_1 vanishes quadratically at the origin (this assumption will be implicitly made throughout the Chapter). Then

$$\alpha_t = \alpha_v + H_t(p, r, \theta)d\theta$$

is a family of almost contact distributions constant along the boundary $F \times \partial\mathbb{D}^2(1)$ such that $\ker \alpha_1$ is a contact structure. The corresponding symplectic structures on $\ker \alpha_t$ is readily constructed as in the previous discussion, and an interpolation to the symplectic form $\alpha_v + dH_1 \wedge d\theta$ is required to obtain the almost contact structure $(\ker \alpha, d\alpha)$. This contact structure does conform property (a) in Theorem 8.1.

The importance of Theorem 8.1 is that it also covers the case of almost contact distributions where H is negative along a part of $F \times \partial\mathbb{D}^2(1)$. This case is handled at the cost of changing the contact structure on $L \times \mathbb{D}^2(1)$. This region is part of the exceptional locus E and should a priori not be modified, however we will see in Section 9 that the control on this region ensured by Theorem 8.1 will be enough to correct that change.

8.1. The model. In this subsection we describe the model used to obtain the contact structure in the statement of Theorem 8.1.

Consider the smooth 5-dimensional manifold $F \times \mathbb{S}^2$. The submanifolds

$$i_0 : F_0 = F \times \{(1,0,0)\} \longrightarrow F \times \mathbb{S}^2$$

$$\text{and } i_\infty : F_\infty = F \times \{(-1,0,0)\} \longrightarrow F \times \mathbb{S}^2$$

are referred to as the fibres at zero and infinity. A construction made relative to F_∞ should be thought as construction on $F \times \mathbb{D}^2(1)$ relative to the boundary.

The compact smooth 3–manifold F is parallelizable. Hence the cotangent bundle $T^*F \longrightarrow F$ is isomorphic to the fibre bundle $F \times \mathbb{R}^3 \longrightarrow F$ given by the projection onto the first factor. The canonical symplectic structure in the manifold T^*F induces a contact structure in the manifold $F \times \mathbb{S}^2$. For instance, given a Riemannian metric the manifold $F \times \mathbb{S}^2$ can be identified with the unit cotangent bundle $\mathbb{S}(T^*F)$ with respect to that metric. This is a convex hypersurface in T^*F and the canonical Liouville vector field defines a contact structure ξ_{can} on $\mathbb{S}(T^*F) \cong F \times \mathbb{S}^2$. The study of the distribution ξ_{can} has been at the core of contact geometry since its foundations. See [**93**] and Appendix 4 in [**4**].

Consider a contact structure (F, Ξ). The choice of a contact form α for Ξ defines an embedding $F \longrightarrow T^*F$. The image of this embedding can be assumed to lie in $\mathbb{S}(T^*F)$. Then (F, Ξ) is seen as a contact submanifold of $(\mathbb{S}(T^*F), \xi_{can})$. The symplectic normal bundle of this contact embedding is isomorphic to Ξ. In particular the embedding has trivial normal bundle if and only if $c_1(\Xi) = 0$. See [**64**] for an application.

The construction of the contact structure in the following Proposition begins with the natural contact structure in $\mathbb{S}(T^*F)$ thought of as a contact structure in the total space of $F \times \mathbb{S}^2 \longrightarrow \mathbb{S}^2$.

In the manifold $\mathbb{S}^1 \times \mathbb{S}^2$ there exists a unique tight contact structure. It is the contact boundary of the symplectic manifold $\mathbb{S}^1 \times \mathbb{D}^3$. The first Chern class of this tight contact structure is $0 \in H^2(\{0\} \times \mathbb{S}^2, \mathbb{Z}) \cong H^2(\mathbb{S}^1 \times \mathbb{S}^2, \mathbb{Z})$. Consider the overtwisted contact structure ξ_{ot} in the homotopy class of plane fields $\{\theta\} \times T\mathbb{S}^2$. It is obtained by performing half Lutz twist in the tight contact structure along the transverse knot $\mathbb{S}^1 \times \{0\}$. This is said to be the standard 2–overtwisted structure on $\mathbb{S}^1 \times \mathbb{S}^2$. Certainly its first Chern class $c_1(\xi_{ot}) = 2$ coincides with $c_1(T\mathbb{S}^2) = 2$. This homotopy class of plane fields is relevant since $T\mathbb{S}^2$ is a horizontal bundle for the projection $\mathbb{S}^1 \times \mathbb{S}^2 \longrightarrow \mathbb{S}^2$.

The basic geometric construction used to prove Theorem 8.1 is the content of the following result. A minor enhancement of the Proposition is also required, it is explained in Corollary 8.2.

PROPOSITION 8.1. *Let (F, ξ_v) be a contact 3–manifold with $c_1(\xi_v) = 0$, $\xi_v = \ker \alpha_v$ and L a transverse link. Consider the manifold $(F \times \mathbb{S}^2, \omega_{\mathbb{S}^2})$ the standard area form on $\mathbb{S}^2$ and the almost contact structure*

$$(\xi, \omega) = (\ker \alpha_v, d\alpha_v + \omega_{\mathbb{S}^2}).$$

Then there exists a contact structure $\xi_f = \ker \alpha_f$ on $F \times \mathbb{S}^2$ conforming the properties:

a. *The contact form α_f restricts to the initial contact form at the fibres F_0 and F_∞:*
$$i_0^* \alpha_f = \alpha_v \text{ and } i_\infty^* \alpha_f = \alpha_v.$$
b. *Consider the inclusion $i_L : L \times \mathbb{S}^2 = \bigsqcup (\mathbb{S}^1 \times \mathbb{S}^2) \longrightarrow F \times \mathbb{S}^2$. Then the contact form $i_L^* \alpha_f$ defines the contact structure ξ_{ot} on each $\mathbb{S}^1 \times \mathbb{S}^2$.*

c. *The almost contact structures (ξ, ω) and $(\ker \alpha_f, d\alpha_f)$ are homotopic relative to F_∞.*

PROOF. This is a rather long proof. It is divided according to the construction and the verification of each of the three properties.

Construction. Since $c_1(\xi_v) = 0$, there exist a global framing $\{X_1, X_2 \in \Gamma(\xi_v)\}$ of the contact distribution ξ_v. Denote by X_0 the Reeb vector field associated to the contact form $\alpha_0 = \alpha_v$. Therefore $\{X_0, X_1, X_2\}$ is a global framing of TF. Let $\{\alpha_0, \alpha_1, \alpha_2\}$ be the dual framing. It can be assumed that the transverse link L is an orbit of the Reeb vector field X_0. In particular α_1 and α_2 vanish along L. Denote the standard embedding of the 2–sphere as $e = (e_0, e_1, e_2) : \mathbb{S}^2 \longrightarrow \mathbb{R}^3$. The previous discussion endows the smooth manifold $F \times \mathbb{S}^2$ with a natural contact structure. We use an explicit model for the argument. It is a computation to verify that

$$\lambda = e_0 \cdot \alpha_0 + e_1 \cdot \alpha_1 + e_2 \cdot \alpha_2$$

is a contact form on $F\times\mathbb{S}^2$. The important properties are that $\{\alpha_0,\alpha_1,\alpha_2\}$ is a framing and the map e is a star–shaped embedding. The contact structure $\ker\lambda$ is contactomorphic to ξ_{can}. From the classical viewpoint it is clear that $\ker\lambda$ is a contact structure. See [**93**].

In spherical coordinates $(t,\theta)\in[0,1]\times[0,1]$ the embedding can be described as

$$\begin{aligned} e_0(t,\theta) &= \cos(\pi t), \\ e_1(t,\theta) &= \sin(\pi t)\cos(2\pi\theta), \\ e_2(t,\theta) &= \sin(\pi t)\sin(2\pi\theta). \end{aligned}$$

Note that $F_\infty = F\times(-1,0,0)$ and $F_0 = F\times(1,0,0)$ are contactomorphic contact submanifolds of $(F\times\mathbb{S}^2,\ker\lambda)$ with trivial normal bundle. Consider two copies of $F\times\mathbb{S}^2$, we can perform a contact fibered sum along their F_∞ fibres, see [**61**]. This operation is done in order to obtain two fibres with the contact form α_0. Those coming from the two zero fibres F_0 in the two copies of $F\times\mathbb{S}^2$. Let us provide an explicit equation for the contact form in this fibered sum.

A tentative modification of λ is obtained by considering the following map

$$\begin{aligned} \kappa_0(t,\theta) &= \cos(2\pi t), \\ \kappa_1(t,\theta) &= \sin(2\pi t)\cos(2\pi\theta), \\ \kappa_2(t,\theta) &= |\sin(2\pi t)|\sin(2\pi\theta), \end{aligned}$$

and the 1–form $\kappa_0\cdot\alpha_0+\kappa_1\cdot\alpha_1+\kappa_2\cdot\alpha_2$. Due to the appearance of the absolute value this form is just continuous. Observe though that in the smooth area it is a contact form. Let us perturb it to a smooth 1–form.

Define a smooth map $t:[0,1]\longrightarrow[0,1]$ such that:

$$t(0)=0,\ t(1/2)=1/2,\ t(1)=1, t'(v)>0 \text{ for } v\in[0,1/2)\cup(1/2,1]$$

$$\text{and } t^{(k)}(1/2)=0\ \forall k\in\mathbb{N}.$$

This allows us to reparametrize the sphere with coordinates $(v,\theta)\in[0,1]\times[0,1]$. The following map is denoted by (e_0,e_1,e_2) in order to ease notation. This should not lead to confusion since the map formerly referred to as (e_0, e_1, e_2) is not to be considered again. Consider the smooth map

$$\begin{aligned} e_0(v,\theta) &= \cos(2\pi t(v)), \\ e_1(v,\theta) &= \sin(2\pi t(v))\cos(2\pi\theta), \\ e_2(v,\theta) &= |\sin(2\pi t(v))|\sin(2\pi\theta). \end{aligned}$$

It is indeed smooth because $t^{(k)}(1/2) = 0$. This almost provides the desired 1–form for the fibre connected sum. Define the smooth function $h(v) = v(1-v)\sin(2\pi v)$ and the 1–form $\eta = c \cdot h(v)d\theta$, where c is a small positive constant.

Assertion. There exists a choice of $c \in \mathbb{R}^+$ such that the 1–form defined as

$$\alpha_f = e_0\alpha_0 + e_1\alpha_1 + e_2\alpha_2 - \eta \tag{8.1}$$

is a contact form over the fibre connected sum of two copies of $F \times \mathbb{S}^2$ along the fibres F_∞.

This concludes the construction of the contact form in the manifold $F \times \mathbb{S}^2$ obtained in the Theorem. The contact form α_f also conforms property a. in the statement of the Theorem.

Proof of Assertion. Consider the following volume form $\nu = \sin(\pi v)dv \wedge d\theta \wedge \alpha_0 \wedge \alpha_1 \wedge \alpha_2$ on $F \times \mathbb{S}^2$ and compute the exterior differential

$$d\alpha_f = de_0 \wedge \alpha_0 + de_1 \wedge \alpha_1 + de_2 \wedge \alpha_2 + e_0 d\alpha_0 + e_1 d\alpha_1 + e_2 d\alpha_2 - d\eta.$$

The contact condition states that $\alpha_f \wedge (d\alpha_f)^2$ is a positive multiple of ν. Let us express it as

$$\alpha_f \wedge (d\alpha_f)^2 = \eta_1 + c\eta_2 + c\eta_3,$$

where η_1, η_2, η_3 are the following 5–forms:

$$\begin{aligned}
\eta_1 &= \begin{vmatrix} e_0 & e_1 & e_2 \\ \partial_t e_0 & \partial_t e_1 & \partial_t e_2 \\ \partial_\theta e_0 & \partial_\theta e_1 & \partial_\theta e_2 \end{vmatrix} t'(v)^2 dv \wedge d\theta \wedge \alpha_0 \wedge \alpha_1 \wedge \alpha_2 = \\
&= 4\pi^2 |\sin(2\pi t(v))| (t'(v))^2 dv \wedge d\theta \wedge \alpha_0 \wedge \alpha_1 \wedge \alpha_2, \\
\eta_2 &= -e_0^2 \cdot h'(v) \cdot \alpha_0 \wedge d\alpha_0 \wedge dv \wedge d\theta, \\
\eta_3 &= -\sum_{i+j\geq 1} (e_i \cdot e_j \cdot h'(v)) \cdot \alpha_i \wedge d\alpha_j \wedge dv \wedge d\theta + \\
&+ \sum_{i,j} (e_i \cdot h(v)) \cdot de_j \wedge d\alpha_i \wedge \alpha_j \wedge d\theta.
\end{aligned}$$

The indices belong to $i, j \in \{0, 1, 2\}$. Evaluating at $v = 1/2$ we obtain:

$$\begin{aligned}
\eta_2(p, 1/2, \theta) &= \frac{\pi}{2} \alpha_0 \wedge d\alpha_0 \wedge dv \wedge d\theta = \frac{\pi}{2} dv \wedge d\theta \wedge \alpha_0 \wedge \alpha_1 \wedge \alpha_2, \\
\eta_1(p, 1/2, \theta) &= 0, \\
\eta_3(p, 1/2, \theta) &= 0.
\end{aligned}$$

Therefore, there is a small constant $\delta > 0$ such that the 5–form $\eta_2 + \eta_3$ is a positive volume form in the region $F \times [1/2 - \delta, 1/2 + \delta] \times [0, 1]$. The function $t(v)$ is strictly increasing except at $v = 1/2$. Hence, there exists a constant $B > 0$ such that $t'(v) > B$ for any $v \in [0, 1/2 - \delta] \cup [1/2 + \delta, 1]$.

Let us write $\eta_1(p, v, \theta) = g_1(p, v, \theta)\nu$ and $\eta_2 + \eta_3 = g_2(p, v, \theta)\nu$. There exist constants $C, M \in \mathbb{R}^+$ such that $g_1 > C > 0$ for $v \in [0, 1/2 - \delta] \cup [1/2 + \delta, 1]$, and $|g_2| \leq M$.

Choose the initial constant $c \in \mathbb{R}^+$ to satisfy $cM \leq C$. Then we obtain the following bound for $v \in [0, 1/2 - \delta] \cup [1/2 + \delta, 1]$:

$$\alpha_f \wedge (d\alpha_f)^2 = \eta_1 + c\eta_2 + c\eta_3 = (g_1 + cg_2)\nu > C - cM \geq 0.$$

Hence the form α_f is a contact form in this region. The following bound holds in the remaining region $v \in [1/2 - \delta, 1/2 + \delta]$:

$$\alpha_f \wedge (d\alpha_f)^2 = \eta_1 + c\eta_2 + c\eta_3 = (g_1 + cg_2)\nu > cg_2 \geq 0.$$

Thus α_f is a contact form in the fibre connected sum $F \times \mathbb{S}^2$. $\square$

Property b. The contact form α_v associated to ξ_v has been chosen such that its Reeb vector field X_0 is tangent to the link L. Thus α_1,α_2 vanish on L. Restricting the contact form α_f in the equation (8.1) to the submanifold we obtain

$$i_L^*(\alpha_T) = \cos(2\pi t(v))dz - cv(1-v)\sin(2\pi v)d\theta, \tag{8.2}$$

where $(z, v, \theta) \in \mathbb{S}^1 \times \mathbb{S}^2$. This is an equation of the contact structure ξ_{ot} on each $\mathbb{S}^1 \times \mathbb{S}^2$. Indeed, consider $a(v) = \cos(2\pi t(v))$ and $b(v) = v(1-v)\sin(2\pi v)$. Then the curve parametrized by $(a(v), b(v))$ rotates once around the origin and the tangent vector field $(a'(t), b'(t))$ is transverse to the radial direction, i.e. ∂_r, on $(0,1)$.

Property c. Let $f_F : F \longrightarrow [0,1]$ be a Morse function on the 3–manifold F with a single minimum $q \in F$. Then

$$f(p, v, \theta) = f_F(p) - (1 + f_F(p))v^2 : F \times \mathbb{S}^2 \longrightarrow [-1, 1]$$

is a smooth Morse–Bott function on $F \times \mathbb{S}^2$ whose non–degenerate critical points belong to the central fibre F_0 and has F_∞ as a critical manifold. Let us use the associated cell decomposition relative to the level $f^{-1}((-\infty, -1]) = F_\infty$. It is generated by the descending manifolds associated to each critical point. It has a unique 2–cell $\sigma_q^2 = \{q\} \times (\mathbb{S}^2 \backslash \{\infty\})$, corresponding to the critical point $(q, 0, 0)$.

Note that the resulting almost contact structure and the initial one coincide near F_∞ and we only need to compare them as almost contact structures on the disk relative to the boundary. Due to Lemma 2.4, a pair of almost contact distributions homotopic over the disk σ_q^2 relative to its boundary are homotopic on the 5–manifold $F \times \mathbb{S}^2$. To conclude Property c. we verify that such relative homotopy exists along σ_q^2. The almost contact distribution ξ in the statement of the Proposition can be written as $\xi = \ker \alpha_v \oplus T\mathbb{S}^2$. Its symplectic structure is induced by the symplectic structure on each of the factors. Note that both $\ker \alpha_v$ and $T\mathbb{S}^2$ are $\mathrm{rk}_\mathbb{R} = 2$ symplectic bundles. This is tantamount to $\mathrm{rk}_\mathbb{R} = 2$ oriented bundles.

Consider a trajectory γ of the Reeb flow through q

$$\gamma : (-\varepsilon, \varepsilon) \longrightarrow F, \quad \gamma(0) = q.$$

The submanifold $(V, \xi_{ot}) = (\gamma \times \mathbb{S}^2, \xi_f|_{\gamma \times \mathbb{S}^2})$ is a contact submanifold of the contact manifold $(F \times \mathbb{S}^2, \ker \alpha_f)$. A contact from is given by the equation (8.2). As suggested by the notation, the contact form $\alpha_{ot} = \alpha_f|_V$ defines the overtwisted structure ξ_{ot} on $(-\varepsilon, \varepsilon) \times (\mathbb{S}^2 \setminus \{\infty\})$.

Hence the two subbundles of TV

$$\xi_{ot} \longrightarrow \sigma_q^2, \quad T\mathbb{S}^2 \longrightarrow \sigma_q^2$$

are homotopic as oriented subbundles relative to the boundary of the disk. Thus relative homotopic as symplectic bundles. This provides a homotopy in the 2–dimensional horizontal part. Let us deal with the vertical bundle.

The initial vertical subbundle is $\xi_v = \ker \alpha_v$, it does satisfy the splitting

$$\xi_{v|\sigma_q^2} \oplus TV_{|\sigma_q^2} = T(F \times \mathbb{S}^2)_{|\sigma_q^2}.$$

The resulting vertical subbundle in the distribution ξ_f can be constructed as the symplectic orthogonal subbundle ν_{ot} of ξ_{ot}. This yields the decomposition

$$\nu_{ot|\sigma_q^2} \oplus TV_{|\sigma_q^2} = T(F \times \mathbb{S}^2)_{|\sigma_q^2}.$$

The space of rank–2 oriented vector bundles transverse to the rank–3 vector bundle TV is contractible. Hence $\nu_{ot|\sigma_q^2}$ is homotopic to $\xi_{v|\sigma_q^2}$ as rank–2 symplectic distributions.

On the unique 2–cell σ_q^2 both splittings $\xi = \xi_v \oplus T\mathbb{S}^2$ and $\xi_f = \nu_{ot} \oplus \xi_{ot}$ hold. Note that the bundle $T\mathbb{S}^2$ is homotopic to ξ_{ot} inside TV and ξ_v is homotopic to ν_{ot} through planes transverse to TV. Since the subbundles are pairwise homotopic as symplectic distributions and these homotopies do not interact, ξ and ξ_f are also homotopic as symplectic distributions. $\square$

In the proof of Property c. of Proposition 8.1 we have only used the 2–skeleton to verify the statement. Lemma 2.4 ensures that this is enough. There is an alternative geometric approach to produce the homotopy. Indeed, the Reeb trajectories of α_v produce a foliation $\mathcal{L}$ on F. This induces a foliation $\mathcal{L} \times \mathbb{D}^2$ with 3–dimensional contact leaves. The argument in the proof of Property c. can be made parametric to construct an explicit almost contact homotopy.

The norm of the function H in the statement of Theorem 8.1 does translate into a geometric feature. This is the size of a certain neighborhood.

This is explained in the subsequent subsection. Let us enhance the conclusion of Proposition 8.1 in order to obtain an arbitrarily large contact neighborhood of a fibre.

Property d. Let $R \in \mathbb{R}^+$ be given. There exists a neighborhood U_∞ of the fibre F_∞ and a trivializing diffeomorphism $\psi : F \times \mathbb{D}^2(R) \longrightarrow U_\infty$ such that

- $\psi(F \times \{0\}) = F_\infty$,
- $\psi^*\alpha_f = \alpha_v + r^2 d\theta$.

This property could have been included in the statement of Proposition 8.1. It is stated apart to ease the comprehension.

COROLLARY 8.2. *There exists a contact manifold $(F \times \mathbb{S}^2, \xi_f = \ker \alpha_f)$ conforming a. to d.*

PROOF. The contact structure $(F \times \mathbb{S}^2, \xi_f = \ker \alpha_f)$ obtained in Proposition 8.1 does satisfy properties a.– c. Let us modify it in order to satisfy Property d. The contact neighborhood theorem provides a neighborhood U_∞ of the fibre F_∞ and a contactomorphism $\psi_\varepsilon : F \times \mathbb{D}^2(\varepsilon) \to U_\infty$, for some $\varepsilon \in \mathbb{R}^+$. In case $R \leq \varepsilon$ the statement follows.

Suppose that $R \geq \varepsilon$, then we use the following covering trick (introduced in [**107**]). Let $k \in \mathbb{N}$ be an integer and consider the ramified covering

$$\begin{aligned} \phi_k : F \times \mathbb{S}^2 = F \times \mathbb{CP}^1 &\longrightarrow F \times \mathbb{CP}^1 \\ (p, z) &\longmapsto (p, z^k). \end{aligned}$$

The branch locus consists of the fibres F_0 and F_∞. Both fibres are contact submanifolds in $(F \times \mathbb{S}^2, \ker \alpha_f)$ and we can lift the contact form to a contact form $\alpha_f^k = \phi_k^* \alpha_f$ in the domain of the covering map. Lifting the formula (8.1), we obtain

$$\alpha_f^k = \cos(2\pi t(v))\alpha_0 + \sin(2\pi t(v))\cos(2\pi k\theta)\alpha_1 + |\sin(2\pi t(v))| \sin(2\pi k\theta)\alpha_2 + k\eta \tag{8.3}$$

The reader can verify that properties a.– c. are still satisfied by the contact structure $\ker \alpha_f^k$. Regarding Property d, observe that $\psi^*\alpha_f^k = \alpha_v + kr^2 d\theta$. Consider the scaling diffeomorphism

$$\begin{aligned} g_k : F \times \mathbb{D}^2(\sqrt{k} \cdot \varepsilon) &\longrightarrow F \times \mathbb{D}^2(\varepsilon) \\ (p, r, \theta) &\longmapsto (p, r/\sqrt{k}, \theta). \end{aligned}$$

Then the trivializing diffeomorphism $\psi_\varepsilon \circ g_k$ satisfies $(\psi_\varepsilon \circ g_k)^* \alpha_f^k = \alpha_v + r^2 d\theta$. Choose $k \in \mathbb{N}$ such that $\sqrt{k} \cdot \varepsilon \geq R$ to conclude the statement. $\square$

To ease notation, we can refer to the contact structures resulting either of Proposition 8.1 or Corollary 8.2 as ξ_f. Since the latter has better properties than the former, ξ_f refers to that in Corollary 8.2.

Remarks 8.3. Suppose that the contact manifold (F, ξ_v) is overtwisted, then the contact structure ξ_f contains a plastikstufe. Confer [**106**],[**113**]. It can be constructed as follows.

Restrict the contact form α_f^k to $\{(p, v, \theta) \in F \times \mathbb{S}^2 : v = 1/2\} \cong F \times \mathbb{S}^1$. This is a contact bundle over the $\mathbb{S}^1$–factor. The induced contact connection satisfies that $\pi^* \partial_\theta = \partial_\theta$ and thus the parallel transport is the identity. In particular, the parallel transport of the overtwisted disk on the fibre generates a plastikstufe.

The contact manifold $F \times \mathbb{D}^2(1/2)$ will be contact embedded in our initial manifold (M, ξ), is PS–overtwisted. Note that Section 6 forces (F, ξ_v) to be overtwisted contact structures. Hence the contact structures constructed in Theorem 1.1 are PS–overtwisted.

8.2. The proof. In this subsection we conclude the proof of 8.1. The essential geometric ideas have been introduced in Proposition 8.1. The necessary details to conclude are provided.

Let us introduce a definition. It is given in order to stress the relevance of the size in a neighborhood.

Definition 8.4. Let $(F, \xi_v = \ker \alpha_v)$ be a contact manifold. For $A \in \mathbb{R}^+$, the manifold $F \times [-A, A] \times \mathbb{S}^1$ with the contact structure $\alpha_A = \alpha_v + t d\theta$ is called the A–standard contact band associated to $(F, \ker \alpha_v)$.

The role of this definition is elucidated in the following lemma.

Lemma 8.5. *Let (F, ξ_F) be a contact manifold, $\xi_F = \ker \alpha_F$. Consider a contact manifold $(F \times [0,1] \times \mathbb{S}^1, \xi)$ with contact form $\alpha_F + H d\theta$, $H \in C^\infty(F \times [0,1] \times \mathbb{S}^1)$.*

Suppose that $|H| < A$, for some $A \in \mathbb{R}^+$. Then, there exists a strict contact embedding of $(F \times [0,1] \times \mathbb{S}^1, \alpha)$ in the A–standard contact band associated to (F, α_F).

PROOF. Consider the embedding defined as

$$\begin{array}{rcl} \Psi_A : F \times [0,1] \times \mathbb{S}^1 & \longrightarrow & F \times [-A, A] \times \mathbb{S}^1 \\ (p,t,\theta) & \longrightarrow & (p, H(p,t,\theta), \theta)\,. \end{array}$$

This is a diffeomorphism onto its image because the form $\alpha_F + Hd\theta$ is a contact form, or equivalently $\partial_t H > 0$. □

The remaining ingredient for the proof of Theorem 8.1 is the subsequent lemma.

Let $l \in \mathbb{R}^+$ be a constant, $l > 1$. Consider a smooth function κ_l : $[0, 2l+1] \longrightarrow [0, l]$ with

$$\kappa_l(r) = 0 \text{ for } r \in [0, l], \quad \kappa_l(r) = r - l - 1 \text{ for } r \in [2l, 2l+1].$$

Consider $(r, \theta) \in \mathbb{D}^2_l$ to be polar coordinates for the 2–disk $\mathbb{D}^2_l$ of radius $2l+1$. Suppose that F is a manifold, the subset $F \times \{a \leq r \leq b\}$ of the product $F \times \mathbb{D}^2$ will be denoted $F \times [a,b] \times \mathbb{S}^1$. Similarly, $F \times (a,b] \times \mathbb{S}^1$ refers to the subset $F \times \{a < r \leq b\} \times \mathbb{S}^1$.

LEMMA 8.6. *Let (F, ξ_v) be a contact 3–manifold with $c_1(\xi_v) = 0$, $\xi_v = \ker \alpha_v$, $l \in (1, \infty)$ and L a transverse link. Consider the standard area $\omega_{\mathbb{D}}$ on the 2–disk $\mathbb{D}^2_l$ and the almost contact structure on $F \times \mathbb{D}^2_l$ described as*

$$(\xi, \omega) = (\ker(\alpha_v + \kappa_l(r)d\theta), d\alpha_v + \omega_{\mathbb{D}}).$$

Then there exists a contact structure $\xi_1 = \ker \alpha_1$ on $F \times \mathbb{D}^2_l$ such that:

A. *The region $F \times [1, 2l+1] \times \mathbb{S}^1$ is an l–standard contact band for $(F, \ker \alpha_v)$:*

$$\alpha_1|_{F \times [1,2l+1] \times \mathbb{S}^1} = \alpha_v + (r - l - 1)d\theta.$$

B. *Consider the inclusion $i_L : L \times \mathbb{D}^2_l = \bigsqcup(\mathbb{S}^1 \times \mathbb{D}^2_l) \longrightarrow F \times \mathbb{D}^2_l$. Then the contact form $i_L^*\alpha_f$ defines a small neighborhood of a full Lutz twist on each $\mathbb{S}^1 \times \mathbb{D}^2_l$.*

C. *(ξ, ω) and $(\xi_1, d\alpha_1)$ are homotopic relative to the boundary $F \times \partial\mathbb{D}^2_l$.*

PROOF. Consider Property d. in Proposition 8.1 and Corollary 8.2 with radius $R = \sqrt{l}$. Let $(F \times \mathbb{S}^2, \xi_f = \ker \alpha_f)$ be the contact manifold obtained in Corollary 8.2. Then there exists a contact neighborhood U_∞ of the fibre F_∞ and a trivializing diffeomorphism

$$\psi : F \times \mathbb{D}^2(\sqrt{l}) \longrightarrow U_\infty \text{ such that } \psi^*\alpha_f = \alpha_v + r^2 d\theta.$$

The diffeomorphism ψ also identifies $\psi : F \times (0, \sqrt{l}] \times \mathbb{S}^1 \longrightarrow U_\infty \setminus F_\infty$.

Define the following map

$$m : F \times [-l, 0) \times \mathbb{S}^1 \longrightarrow F \times (0, \sqrt{l}] \times \mathbb{S}^1, \quad m(p, x, \theta) = (p, \sqrt{-x}, -\theta).$$

It satisfies $(\psi \circ m)^*\alpha_f = \alpha_v + rd\theta$. This form extends to the region $F \times [-l, l] \times \mathbb{S}^1$ with the same expression. Then the manifold $F \times \mathbb{D}^2_l$ is obtained by gluing the annular region $F \times [0, l] \times \mathbb{S}^1$ to the annular region

$$F \times (0, \sqrt{l}] \times \mathbb{S}^1 \cong F \times [-l, 0) \times \mathbb{S}^1 \text{ identified via } m,$$

and using the contactomorphism ψ restricted to $F \times (0, \sqrt{l}] \times \mathbb{S}^1$ to perform the gluing construction in $(F \times \mathbb{S}^2) \setminus F_\infty$. The construction implies that Property A holds. Properties B and C follow from Properties b and c in Corollary 8.2 since the manifold $(F \times \mathbb{S}^2) \setminus F_\infty$ satisfies them. $\square$

Proof of Theorem 8.1. Let $\varepsilon > 0$ be a small constant. The function H is C^0–bounded on the compact manifold $\mathcal{F} = F \times \mathbb{D}^2(1)$. Let $l \in (1, \infty)$ be an upper bound such that $\|H\|_{C^0} < l - \varepsilon/4$. Consider coordinates $(p, r, \theta) \in \mathcal{F}$ and a smooth function $h \in C^\infty(\mathcal{F})$ such that

- $h(p, r, \theta) = 0$ for $r \in [0, 1 - 2\varepsilon]$,
- $h(p, r, \theta) = r - l - (1 - \varepsilon)$ for $r \in [1 - \varepsilon, 1 - 3\varepsilon/4]$,
- $\partial_r h > 0$ for $r \in [1 - 3\varepsilon/4, 1 - \varepsilon/2]$,
- $h(p, r, \theta) = H(p, r, \theta)$ for $r \in [1 - \varepsilon/2, 1]$.

The almost contact structure (ξ, ω) is homotopic relative to the boundary to the almost contact structure defined by

$$(\xi_h, \omega_h) = (\ker(\alpha_v + h(p, r, \theta)), d\alpha_v + (1 - \tau(r)) \cdot rdr \wedge d\theta + \tau(r)dh \wedge d\theta).$$

The homotopy is provided by a relative homotopy between the functions $h(p, r, \theta)$ and $H(p, r, \theta)$ and Lemma 2.5. Hence the departing almost contact structure can be considered to be (ξ_h, ω_h) .

The neighborhood $F \times (1 - \varepsilon, 1] \times \mathbb{S}^1$ of the boundary $F \times \partial\mathbb{D}^2(1) \subset \mathcal{F}$ is a contact manifold. By Lemma 8.5, $F \times (1 - \varepsilon, 1] \times \mathbb{S}^1$ contact embeds in an l–standard contact band $F \times [-l, l] \times \mathbb{S}^1$. Denote this embedding by ϕ. It depends on the Hamiltonian $h \in C^\infty(\mathcal{F})$ in the interval $(1 - \varepsilon, 1]$. Observe that $\phi(F \times \{1 - \varepsilon\} \times \mathbb{S}^1) = F \times \{-l\} \times \mathbb{S}^1$ since $h(p, 1 - \varepsilon, \theta) = -l$.

Consider the almost contact manifold $(F \times \mathbb{D}^2_l, \xi_1 = \ker \alpha_1)$ in the statement of Lemma 8.6. Property A implies the existence of a contactomorphism

$$\iota : F \times [-l, l] \times \mathbb{S}^1 \longrightarrow F \times [1, 2l + 1] \times \mathbb{S}^1 \subset (F \times \mathbb{D}^2_l, \xi_1),$$
$$\iota(p, r, \theta) = (p, r + (l + 1), \theta)$$

embedding the l–standard contact band in a neighborhood of size $2l$ of the boundary of $F \times \mathbb{D}^2_l$. Consider the composition

$$j = \iota \circ \phi : F \times (1 - \varepsilon, 1] \times \mathbb{S}^1 \longrightarrow F \times \mathbb{D}^2_l.$$

In particular it satisfies $j(F \times \{1 - \varepsilon\} \times \mathbb{S}^1) = F \times \{1\} \times \mathbb{S}^1 \subset F \times \mathbb{D}^2_l$ and embeds a neighborhood of the boundary $F \times \{1 - \varepsilon\} \times \mathbb{S}^1$ via

$$j : F \times (1 - \varepsilon, 1 - 7\varepsilon/8) \times \mathbb{S}^1 \subset \mathcal{F} \longrightarrow F \times [1, 2l + 1] \times \mathbb{S}^1 \subset F \times \mathbb{D}^2_l,$$
$$j(p, r, \theta) = (p, r + \varepsilon, \theta).$$

The required contact structure in the statement of Theorem 8.1 is obtained by extending j to the interior of the manifold $F \times \mathbb{D}^2(1-\varepsilon) \subset \mathcal{F}$ and pulling–back the contact structure from $(F \times \mathbb{D}^2_l, \ker \alpha_1)$. Indeed, consider $\widetilde{j}$ a smooth embedding such that

$$\widetilde{j} : F \times \mathbb{D}^2(1) \longrightarrow F \times \mathbb{D}^2_l, \quad \widetilde{j}|_{F \times (\mathbb{D}^2(1) \setminus \mathbb{D}^2(1-\varepsilon))} = j.$$

For instance one can consider the extension to be

$$\widetilde{j}|_{F \times \mathbb{D}^2(1-\varepsilon)} : F \times \mathbb{D}^2(1 - \varepsilon) \longrightarrow F \times \mathbb{D}^2(1), \quad (p, r, \theta) \longmapsto (p, c(r), \theta),$$

where $c : [0, 1 - \varepsilon] \longrightarrow [0, 1]$ is a smooth function such that

- $c(t) = t$ near $t = 0$,
- $c(t) = t + \varepsilon$ near $t = 1 - \varepsilon$,
- $c'(t) > 0$ for $t \in [0, 1]$.

Then $\widetilde{j}^*(\xi_1)$ is the required contact structure. Property B in Lemma 8.6 and the fact that the function H is positive in a neighborhood of L imply Property b in the Theorem.

Let us justify that the obtained contact structure is homotopic to the initial almost contact structure relative to the boundary $F \times \partial\mathbb{D}^2(1)$. The homotopy obstruction appears in the 2–skeleton and therefore it is enough to find the homotopy at a disk $\{p\} \times \mathbb{D}^2(1) \subset \mathcal{F}$. An analogous computation to the one detailed in the proof of Property c. of Proposition 8.1 yields the same result. Hence the resulting contact structure ξ_1 is homotopic as an almost contact structure to the initial almost contact structure (ξ, ω) relative to the boundary. $\square$

REMARK 8.7. The central ingredient in this construction is the existence of a contact structure ξ on $F \times \mathbb{S}^2$ with the following two properties:

- It restricts to a given contact structure (F, ξ_F) on a fibre $F \times \{p\}$,
- The contact structure ξ is homotopic to the almost contact structure $\xi_F \oplus T\mathbb{S}^2$.

The use of the space of contact elements space forces the fibre to have vanishing Chern class and part of Section 5 is invested to achieve this hypothesis. Since the submission of this Chapter, the articles [**16, 76**] provide a contact structure on $F \times \mathbb{S}^2$ conforming the above properties. Their use would simplify Subsection 5.3.

9. Horizontal Deformation II

The arguments in the previous sections are gathered to conclude the proof of Theorem 1.1.

9.1. Contact Structure in the fibration.

THEOREM 9.1. *Let (M, ξ, ω) be an almost contact structure and (f, C, E) a good ace fibration adapted to it. Suppose that (ξ, ω) is vertical with respect to (f, C) and T is an adapted family such that ξ is a contact structure over a regular neighborhood of $|T|$. Then (ξ, ω) is homotopic to a contact structure ξ' and the restriction of ξ' to the exceptional 3–spheres in E induces the homotopically standard overtwisted contact structure.*

The standard overtwisted structure is the unique overtwisted contact structure on $\mathbb{S}^3$ homotopic to the standard contact structure ξ_{std}.

A neighborhood of the intersection of an exceptional 3–sphere with a fibre of f is diffeomorphic to $\mathbb{S}^1 \times \mathbb{D}^2 \times \mathbb{D}^2$. Let $(z, r, \theta, \rho, \phi)$ be coordinates for such a neighborhood, the triple (z, ρ, ϕ) belong to the fibre. It can be considered as a trivial fibration over the first pair of factors

$$\pi : \mathbb{S}^1 \times \mathbb{D}^2 \times \mathbb{D}^2 \longrightarrow \mathbb{S}^1 \times \mathbb{D}^2, \quad (z, r, \theta, \rho, \phi) \longmapsto (z, r, \theta).$$

There also exists a contact structure given by the contact form $\alpha = dz + r^2 d\theta + \rho^2 d\phi$ on the neighborhood. This induces a contact connection A_π for the fibration π. Let $\delta \in \mathbb{R}^+$ and suppose the horizontal 2–disk $(\rho, \phi) \in \mathbb{D}^2(\delta)$ is of radius δ.

LEMMA 9.1. *Consider the contact manifold $(\mathbb{S}^1 \times \mathbb{D}^2 \times \mathbb{D}^2(\delta), \ker(dz + r^2 d\theta + \rho^2 d\phi))$, π the projection onto the first pair of factors and A_π the* *associated contact connection. The flow of the lift of* ∂_r *to* A_π *preserves the submanifold* $\{(z, r, \theta, \rho, \phi) \in X : \rho = \delta/2\}$.

Proof. The vector field ∂_r belongs to the contact distribution. The vertical directions are generated by $\partial_\rho, \partial_\phi$ and the symplectic form pairs them via $\rho \cdot d\rho \wedge d\phi$. Hence ∂_r is itself the lift to A_π. The statement follows. □

Proof of Theorem 9.1. The complement of a regular neighborhood of $|T|$ in $\mathbb{CP}^1$ is a disjoint collection $\{B_1, \dots, B_a\}$ of 2–disks. The distribution ξ is a contact structure in the fibres of f close to the boundary of $B_1 \cup \dots \cup B_a$. The restriction of f to the preimages of each $\mathcal{B} \in \{B_i\}$ is a smooth fibration since the critical values of f lie in the complement of the set $B_1 \cup \dots \cup B_a$. In order to conclude the statement of the Theorem we produce a deformation over each ball $\mathcal{B}$ supported away from the boundary and resulting in a contact structure.

The proof of the statement now uses the results in Section 8. Let us precise the necessary details regarding the trivializations. Choose a ball $\mathcal{B} \in \{B_1, \dots, B_a\}$ and a local chart $\varphi : \mathcal{B} \longrightarrow B^2(1)$. Consider the map $g = \varphi \circ f : f^{-1}(\mathcal{B}) \longrightarrow B^2(1)$. For $\varepsilon > 0$ a small constant, we may assume that $g^{-1}(B^2(1) \backslash B^2(1-\varepsilon))$ is an open set where the distribution ξ is a contact structure.

Consider an exceptional divisor E. According to the local model used in Section 5, there exists a neighborhood $\mathcal{E}$ of E and a contactomorphism

$$\varphi_E : (\mathbb{S}^3 \times \mathbb{D}^2(\delta), \alpha_{std} + \rho^2 d\phi) \longrightarrow \mathcal{E}.$$

The composition $f \circ \varphi_E : \mathbb{S}^3 \times \mathbb{D}^2(\delta) \longrightarrow \mathbb{S}^2$ restricts to the Hopf fibration at $\mathbb{S}^3 \times \{0\}$. Restricting to the region $f^{-1}(\mathcal{B}) \cap \mathcal{E}$ we obtain a fibration

$$\varphi \circ f \circ \varphi_E : \mathbb{S}^1 \times B^2(1) \times \mathbb{D}^2(\delta) \longrightarrow B^2(1)$$

over the 2–ball. Lemma 2.3 implies that the contact parallel transport along the neighborhoods of the boundary is tangent to it. Lemma 7.3 allows us to radially trivialize and express the contact structure as

$$\xi = \ker(\alpha_v + H d\theta).$$

Observe that the contact fibration is a contact structure in the neighborhood $\mathcal{E}$, therefore $\partial_r H \geq 0$ is satisfied on $\mathcal{E}$. Since $H(p, 0, 0) = 0$, we also conclude that $H \geq 0$ over $\mathcal{E}$.

This setup satisfies the hypotheses of Theorem 8.1. It applies producing a homotopy ξ_t of almost contact structures over $f^{-1}(\mathcal{B})$ relative to its boundary such that $\xi_0 = \xi$ and ξ_1 is a contact structure. The exceptional divisors are contact submanifolds of ξ_1 and their induced contact structure is the standard contact structure ξ_{std} with a full Lutz twist performed. The construction is made relative to the pre–image of a neighborhood of the boundary of the ball $\mathcal{B}$. The argument successively applies to the elements of $\{B_1, \dots, B_a\}$. This concludes the statement. $\square$

9.2. Interpolation at the exceptional divisors. Let (M, ξ, ω) be an almost contact manifold. The argument for proving Theorem 1.1 begins with a good almost contact pencil (f, C, E). Section 5 provides a good ace fibration in a modified manifold $(\widetilde{M}, \widetilde{\xi}, \widetilde{\omega})$. The results in Sections 6, 7 and 8 confer good ace fibrations. These exist not on the manifold (M, ξ, ω) but in $(\widetilde{M}, \widetilde{\xi}, \widetilde{\omega})$. In the previous subsection a contact structure has been obtained in the almost contact manifold $(\widetilde{M}, \widetilde{\xi}, \widetilde{\omega})$ such that a neighborhood of the exceptional spheres has remained contact. It is left to obtain a contact structure in the initial manifold M.

The exceptional spheres in $(\widetilde{M}, \widetilde{\xi})$ have the standard tight contact structure $(\mathbb{S}^3, \xi_{std})$ at the beginning of the argument. In the deformation performed in Section 8 the exceptional spheres become overtwisted and we cannot directly obtain a contact structure on M. This has a simple solution, we deform the contact distribution on a neighborhood of the exceptional spheres to the standard one. This is the content of the following

Theorem 9.2. *Let $(\mathbb{S}^3 \times B^2(4), \xi_0)$ have the contact form*

$$\eta = \alpha_{ot} + \delta \cdot r^2 d\theta, \tag{9.1}$$

where $\delta \in \mathbb{R}^+$ is a constant and α_{ot} is any contact form associated to an overtwisted contact structure homotopic to the standard contact structure on $\mathbb{S}^3$.

Let ξ_{std} be a tight contact structure on $\mathbb{S}^3$. Then there exists a deformation ξ_1 of ξ_0 supported in $\mathbb{S}^3 \times B^2(3)$ such that the ξ_1 is a contact structure and $\mathbb{S}^3 \times \{0\}$ inherits the contact structure ξ_{std}.

This result is a consequence of Lemma 3.2 in [**56**]. Let us give an alternative argument, pointed out to us by Y. Eliashberg.

Proof of Theorem 9.2. Let us begin with the tight contact structure on the 3–sphere $(\mathbb{S}^3, \xi_{std})$. Performing a Lutz twist along a given transverse trivial knot K produces an overtwisted contact structure ξ^1_{ot} in $\mathbb{S}^3$ homotopic to ξ_{std} as an almost contact distribution. The contact structure ξ^1_{ot} is isotopic to the contact structure $\xi^2_{ot} = \ker \alpha_{ot}$. Consider both a trivial Legendrian knot $L \subset (\mathbb{S}^3, \xi_{std})$ whose positive transverse push–off is K, and its Legendrian push–off L' with two additional zig–zags. According to [**41**] a Lutz twist along K is tantamount to a contact $(+1)$–surgery along L and L'. Hence, given $(\mathbb{S}^3, \xi^1_{ot})$ there exists a (-1)–surgery on $(\mathbb{S}^3, \xi^1_{ot})$ producing $(\mathbb{S}^3, \xi_{std})$. Such surgery provides a Liouville cobordism (W, λ) from $(\mathbb{S}^3, \xi^1_{ot})$ to $(\mathbb{S}^3, \xi_{std})$.

The cobordism obtained by a $(+1)$–surgery along L and L' can be made smoothly trivial, see [**41**]. Consider $\theta \in \mathbb{S}^1$ and $\eta^1 = \lambda + \mu \cdot d\theta$, for a constant $\mu \in \mathbb{R}^+$. Then the contactization $(W \times \mathbb{S}^1, \eta^1)$ of the exact symplectic manifold $(W, \lambda) \cong (\mathbb{S}^3 \times [0,1], \lambda)$ is diffeomorphic to $\mathbb{S}^3 \times [0,1] \times \mathbb{S}^1$. We have obtained a contact structure on the 3–sphere times the annulus such that the inner boundary $\mathbb{S}^3 \times \{0\}$ has fibres $(\mathbb{S}^3, \xi_{std})$, and $(\mathbb{S}^3, \xi^1_{ot})$ are the fibres of the outer bundary $\mathbb{S}^3 \times \{1\}$. The inner part is a convex boundary and it can be filled with the contact manifold

$$(\mathbb{S}^3 \times \mathbb{D}^2, \ker(\alpha_{std} + r^2 d\theta))$$

in order to obtain a contact structure on $\mathbb{S}^3 \times \mathbb{D}^2$ with $(\mathbb{S}^3, \xi_{std})$ as central fibre. For a choice of μ small enough, there exists a small constant $\delta \in \mathbb{R}^+$ such that in a neighborhood $\mathbb{S}^3 \times (1-\varepsilon, 1] \times \mathbb{S}^1$ of the outer boundary the contact structure can be expressed as

$$\eta^1 = \alpha^1_{ot} + \delta \cdot r^2 d\theta.$$

The contact forms α^1_{ot} and $\alpha^2_{ot} = \alpha_{ot}$ are isotopic via a family of contact forms $\{\alpha^r_{ot}\}$, $r \in [1,2]$. On the manifold $\mathbb{S}^3 \times [1,4] \times \mathbb{S}^1$ consider the 1–form

$$\eta^2 = \widetilde{\alpha}_{ot} + \delta \cdot r^2 d\theta \text{ for } r \in [1,2] \text{ and } \eta^2 = \alpha^2_{ot} + \delta \cdot r^2 d\theta \text{ for } r \in [2,4]$$

where $\widetilde{\alpha}_{ot}(p, r, \theta) = \alpha^r_{ot}(p)$. The form η^2 is a contact form because the form $r^2 d\theta$ does not depend on the point $p \in \mathbb{S}^3$. The gluing of the contact forms η^1 and η^2 is the required contact structure ξ_1 on $\mathbb{S}^3 \times B^2(4)$. $\square$

Notice that this deformation gives a homotopy of almost contact structures.

9.3. Proof of Theorem 1.1. Let (M, ξ, ω) be an almost contact structure. Applying Lemma 2.2 we suppose that (ξ, ω) is an exact quasi–contact structure. Proposition 5.6 allows us to construct a good almost contact pencil for an homotopic almost contact structure also referred to as (ξ, ω). Then Theorem 4.1 provides a good ace fibration (f, C, E) on an almost contact manifold $(\widetilde{M}, \widetilde{\xi}, \widetilde{\omega})$, a contact neighborhood $\mathcal{N}(B)$ of B and a diffeomorphism $\Pi : \widetilde{M} \backslash E \longrightarrow M \backslash B$ such that $(\Pi_* \widetilde{\xi}, \Pi_* \widetilde{\omega}) = (\xi, \omega)$.

Theorems 6.1, 7.1 and 9.1 subsequently applied to this almost contact manifold and good ace fibration yield a contact structure $\widetilde{\xi}_c$ on $\widetilde{M}$. It induces the standard overtwisted structure on the exceptional spheres since a sequence of full Lutz twists are performed. Apply Theorem 9.2 to deform the contact structure to be the initial tight contact structure near each of the exceptional spheres. Then, maybe after a small deformation, it coincides with $(\widetilde{\xi}, \widetilde{\omega})$ in a tubular neighborhood $\mathcal{N}(E)$ of E. Let us still refer to this contact structure as $\widetilde{\xi}_c$. The distribution $\Pi_* \widetilde{\xi}_c$ defines a contact structure on $M \setminus \mathcal{N}(B)$. It coincides with $(\Pi_* \widetilde{\xi}, \Pi_* \widetilde{\omega}) = (\xi, \omega)$ in the submanifold $\Pi(\mathcal{N}(E) \setminus E)$. The almost contact structure (ξ, ω) is a contact structure in a neighborhood of $\mathcal{N}(B)$. In consequence $\Pi_* \widetilde{\xi}_c$ can be extended to a contact structure ξ_c on M. This concludes the proof of the existence of a contact structure ξ_c in the manifold M.

Let us prove that ξ and ξ_c are homotopic. There exists a homotopy between $(\widetilde{\xi}, \widetilde{\omega})$ and $\widetilde{\xi}_c$ over $\widetilde{M}$. This homotopy restricts to a homotopy over the open submanifold $\widetilde{M} \setminus E$. Then, the diffeomorphism Π yields a homotopy between (ξ, ω) and ξ_c in the open manifold $M \setminus \mathcal{N}(B)$. Let us consider a cell decomposition of the manifold M such that $\mathcal{N}(B)$ does not intersect the 2–skeleton. Such decomposition exists because B is 1–dimensional, M is 5–dimensional and the genericity of transversality. Thus (ξ, ω) and ξ_c are homotopic over the 2–skeleton of this cell decomposition. Then Lemma 2.4 implies that the almost contact structures (ξ, ω) and ξ_c are also homotopic over M. ☐

9.4. Uniqueness. The uniqueness of a contact structure in every homotopy class of almost contact structures does not hold in a 5–manifold. There are many examples in the literature, for instance [**113**] provides two non–contactomorphic contact structures in the same almost contact homotopy class.

The construction described in this Chapter requires a fair amount of choices. Though, the dependence of the contact structure with respect to them may be understood. The three main ingredients are the stabilization procedure of almost contact pencils, in the same spirit than Giroux's stabilization for a contact open book decomposition [**37, 87**], the addition of fake curves in the triangulation increasing the amount of holes filled with the local model and the surgery procedure.

10. Non–coorientable case

10.1. Definitions. Let M be a $(2n+1)$–dimensional closed manifold, not necessarily orientable. In order to state the Theorem 1.1 in the non–coorientable setting, we need to give a definition of a non–coorientable almost contact structure. This is a distribution with a suitable reduction of the structure group along with a property requiring a relation between the normal bundle and the distribution. First we introduce the Lie group $\mathfrak{A}(n)$ defined as

$$\mathfrak{A}(n) = \{A \in O(2n) : \quad AJ = \pm JA\}, \quad \text{where } J = \begin{pmatrix} 0 & Id_n \\ -Id_n & 0 \end{pmatrix}$$

Notice the following properties:

1. The group $\mathfrak{A}(n)$ has two connected components. It is homeomorphic to $U(n) \times \mathbb{Z}_2$.
2. Its group structure is isomorphic to a semidirect product $U(n) \rtimes_\rho \mathbb{Z}_2$. More precisely, let $\mathbb{I} = \begin{pmatrix} Id_n & 0 \\ 0 & -Id_n \end{pmatrix}$, then the action
$$\rho : \mathbb{Z}_2 \longrightarrow Aut(U(n)), \quad a \longmapsto (U \longmapsto \mathbb{I}^a U \mathbb{I}^a)$$
induces the semidirect product structure in the usual way.
3. There is a natural group morphism $\mathfrak{s} : \mathfrak{A}(n) \longrightarrow \mathbb{Z}_2$ defined as
$$\mathfrak{s}(A) = tr(JAJ^{-1}A^{-1})/(2n),$$
i.e. under the previous isomorphism, $\mathfrak{s}$ is the projection onto the second factor of $U(n) \rtimes_\rho \mathbb{Z}_2$.

Let us deduce some topological implications of the existence of a contact structure. Let $\xi \subset TM$ be a possibly non–coorientable contact structure on M with a fixed set $\{U_i\}$ of trivializing contractible charts. Choose α_i

as a local equation for $\xi|_{U_i}$, then

$$\alpha_i = a_{ij}\alpha_j, \qquad \text{with } a_{ij} : U_i \cap U_j \longrightarrow \{\pm 1\}.$$

This implies that $\{a_{ij}\}$ are the transition function of the normal line bundle TM/ξ. Further, $(d\alpha_i)_{|\xi} = a_{ij}(d\alpha_j)_{|\xi}$. In particular, we may choose a family of compatible complex structures $\{J_i\}$ for the bundle ξ satisfying $J_i = a_{ij}J_j$.

First, note that there is a group injection

$$\mathfrak{A}(n) \longrightarrow O(2n+1), \quad A \longmapsto \begin{pmatrix} A & 0 \\ 0 & \mathfrak{s}(A) \end{pmatrix}$$

and thus the structure group of M reduces to $\mathfrak{A}(n)$. And second, a $\mathfrak{A}(n)$–bundle E induces via the morphism $\mathfrak{s}$ a real line bundle $\mathfrak{s}(E)$. This construction applied to ξ gives the line bundle TM/ξ in the case above. These two properties will be the ones required in the following:

Definition 10.1. An almost contact structure on a manifold M is a codimension 1 distribution $\xi \subset TM$ such that the structure group of ξ reduces to $\mathfrak{A}(n)$ and $\mathfrak{s}(\xi) \cong TM/\xi$.

Observe that the definition for a cooriented almost contact distribution coincides with the one previously given. There are some immediate topological consequences of the existence of such a ξ. Indeed:

(1) If n is an even integer, then $\mathfrak{A}(n) \subset SO(2n)$. Thus the distribution ξ is oriented.
(2) If n is an even integer, there is an isomorphism

$$TM/\xi \cong det(TM). \tag{10.1}$$

Hence, any almost contact structure in an orientable 5–dimensional manifold is cooriented. Conversely, any non–orientable 5–manifold can only admit non–corientable almost contact structures.
(3) If n is an odd integer, then $\mathfrak{s} = det$ as morphisms from $\mathfrak{A}(n)$ to $\mathbb{Z}_2$. Therefore M is orientable since

$$det(TM) \cong det(\xi \oplus (TM/\xi)) \cong det(\xi) \otimes \mathfrak{s}(\xi) \cong det(\xi)^2 \cong \mathbb{R}$$

Let M^{2n+1} be a non–orientable manifold with n an even integer. Then there exists a canonical $2:1$ cover

$$\pi_2 : M_2 \longrightarrow M$$

satisfying the following properties:

1. M_2 is an orientable manifold.
2. Any almost contact structure ξ on M lifts to an almost contact structure $\pi_2^*\xi$ on M_2. Moreover, such a distribution is cooriented because of equation (10.1).

10.2. Statement of the main result. Let us state the equivalent of Theorem 1.1 in the non–coorientable setting:

THEOREM 10.1. *Let M be a non–orientable closed 5–dimensional manifold. Let ξ be an almost contact structure. Then there exists a contact structure ξ_c homotopic to ξ.*

PROOF. Let $\pi_2 : (M_2, \pi_2^*\xi) \longrightarrow (M, \xi)$ be an orientable double cover. The constructions developed in this Chapter can be performed in a $\mathbb{Z}_2$–invariant manner. Let us discuss it:

(1) An almost contact pencil (f, B, C) can be made $\mathbb{Z}_2$–invariant. To be precise, the loci B and C are $\mathbb{Z}_2$–invariant subsets and f is a $\mathbb{Z}_2$–invariant as a map. In particular the action preserves the fibres. This is because the approximately holomorphic techniques can be developed in that setting. See [**83**] for the details of the construction in the $\mathbb{Z}_2$–invariant setting.

(2) The deformations performed in Section 4 can easily be done in a $\mathbb{Z}_2$–invariant way. Also, the surgery along a $\mathbb{Z}_2$–invariant loop can be built to preserve that symmetry.

(3) Subsection 6.2 is also prepared for the $\mathbb{Z}_2$–invariant setting. Instead of having a single pair of overtwisted disks, we require two pairs of overtwisted disks. Each pair in the image of the other through the $\mathbb{Z}_2$–action.

(4) Eliashberg's construction is not $\mathbb{Z}_2$–invariant. Therefore we proceed by quotienting the whole manifold by the $\mathbb{Z}_2$–action, we then obtain an almost contact pencil over the quotient. The fibres are oriented since they are 3–dimensional almost contact manifolds. The induced almost contact distribution on them is non–coorientable. However, there is no hypothesis on the coorientability in the results of [**45**]. Once the procedure described in Section 6 is applied, we consider the orienting double cover.

(5) Section 7 is trivially adapted to the $\mathbb{Z}_2$–invariant setting if a serious increase of notation is allowed.

(6) Filling the 2–cells as in Section 8 and 9. We need to produce a $\mathbb{Z}_2$–invariant standard model over $M \times \mathbb{S}^2$, with (M, α_0) a contact manifold with a $\mathbb{Z}_2$–invariant action. The only required ingredient is to ensuring that the framing $\{\alpha_0, \alpha_1, \alpha_2\}$ is chosen $\mathbb{Z}_2$–invariant. The rest of the proof works through up to notation details.

(7) The arguments in Section 9 are still $\mathbb{Z}_2$–invariant if the previous choices have been done $\mathbb{Z}_2$–invariantly. Therefore, we obtain a $\mathbb{Z}_2$–invariant contact structure ξ_c^2 on M_2. Its quotient produces a contact structure ξ_c on M.

This proves the existence part of the statement. The statement concerning the homotopy follows since the homotopies can be easily made $\mathbb{Z}_2$–invariant. $\square$

CHAPTER 3

Geometric criteria for overtwistedness

In this third chapter we establish geometric criteria to decide whether a contact manifold is overtwisted. Starting with the original definition from [**15**], we first relate the different overtwisted disks (D^{2n}, ξ_{ot}) in each dimension and show that a manifold is overtwisted if the Legendrian unknot is loose. Then we characterize overtwistedness in terms of open book decompositions and provide several applications. This is joint work with E. Murphy and F. Presas.

1. Introduction

A contact structure on a $(2n-1)$–dimensional smooth manifold Y is a maximally non–integrable codimension 1 distribution ξ. Recently a special class of contact structures has been introduced [**15**] in any dimension: the overtwisted contact structures. Generalizing the original definition and results in the 3-dimensional case [**45**], it is shown in [**15**] that overtwisted contact structures satisfy a parametric h-principle, i.e. their classification up to isotopy coincides with the classification of homotopy classes of almost contact structures. This classification then becomes a strictly algebraic topological problem which can be solved via obstruction theory. The definition of overtwisted contact structures is reviewed in Section 2.

1.1. The main theorem. Though the result in [**15**] demonstrates the existence of overtwisted contact structures homotopic to any almost contact structure, a significant drawback to the existence proof is that the construction is fairly non–explicit. There is a lack of examples of closed overtwisted contact manifolds of dimension $2n-1>3$, and the techniques used in [**15**] give no criterion in order to show that a given manifold is overtwisted, other than an direct application of the definition. The main result of this paper gives a number of equivalent conditions for overtwistedness.

THEOREM 1.1. *Let (Y, ξ) be a contact manifold of dimension $2n-1>3$. Choose a contact form α_{ot} on $\mathbb{R}^3$ which defines an overtwisted contact* *structure. Then there is a constant $R \in \mathbb{R}^+$ depending only on α_{ot} and n, so that the following conditions are equivalent:*

1. *(Y, ξ) is overtwisted.*
2a. *$\exists$ contact embedding of $(\mathbb{R}^3 \times \mathbb{C}^{n-2}, \ker\{\alpha_{ot} + \lambda_{st}\})$ into (Y, ξ).*
2b. *$\exists$ contact embedding of $(\mathbb{R}^3 \times D^{2n-4}(R), \ker\{\alpha_{ot} + \lambda_{st}\})$ into (Y, ξ).*
3a. *The standard Legendrian unknot $\Lambda_0 \subseteq Y$ is loose.*
3b. *(Y, ξ) contains a small plastikstufe with spherical core and trivial rotation.*
4. *There is an open book compatible with (Y, ξ) which is a negative stabilization.*

In the statement of Theorem 1.1 above, λ_{st} is the standard Liouville form on both the disk $D^{2n-4}(R)$ and $\mathbb{C}^{n-2}$, defined by

$$\lambda_{\mathrm{st}} = \frac{1}{2}\sum_{i=1}^{n-2}(x_i dy_i - y_i dx_i) = \frac{1}{2}\sum_{i=1}^{n-2} r_i^2 d\theta_i.$$

The *standard Legendrian unknot* $\Lambda_0 \subseteq (\mathbb{R}^{2n-1}, \xi_0)$ is defined to be

$$\Lambda_0 = \{y_i = 0 : i = 1, \ldots, n\} \cap S^{2n-1} \subseteq (\mathbb{R}^{2n-1}, \xi_0) = (S^{2n-1}, \xi_0) \setminus \{\text{point}\} \subseteq \mathbb{C}^n.$$

The standard Legendrian unknot $\Lambda_0 \subseteq (Y, \xi)$ is defined by the inclusion of a Darboux chart $\Lambda_0 \subseteq (\mathbb{R}^{2n-1}, \xi_0) \subseteq (Y, \xi)$, all of which are isotopic. The concept of *loose* Legendrians were first studied in [**104**], and see also [**36, 105**].

The *plastikstufe* is an n-dimensional submanifold $\mathcal{P} \subseteq (Y, \xi)$ so that the contact structure ξ is equivalent to $D^2_{\mathrm{ot}} \times \{p = 0\} \subseteq \mathbb{R}^3_{\mathrm{ot}} \times T^*Q$ near $\mathcal{P}$, where Q is a closed manifold called the *core* of $\mathcal{P}$. The plastikstufe was first defined in [**106**] and shown there to be an obstruction to symplectic fillability. See also [**105**] for the definitions of "small" and "trivial rotation", and how the plastikstufe relates to loose Legendrians.

Open books which are compatible with (Y, ξ) appear in the Giroux correspondance between open books and contact structures [**37, 71**]. In order to for (4) in Theorem 1.1 and this discussion to apply, we suppose that (Y, ξ) is closed. An appropriate open book decomposition of the manifold Y determines a contact structure ξ, and every contact manifold (Y, ξ) admits such an adapted open book decomposition. This adapted open book can be positively or negatively stabilized. The resulting open books induce two contact structures ξ_+ and ξ_- on Y. The positive stabilization (Y, ξ_+) is contactomorphic to (Y, ξ), but (Y, ξ_-) typically is not. In particular, the negative stabilization of a contact structure was known to have vanishing symplectic field theory [**21, 22**], Theorem 1.1 provides a different proof of this fact.

1.2. Consequences A. Theorem 1.1 and the methods developed in its proof can be used to deduce several results. Let us begin by listing two of them.

Given a contact 3–fold (Y, ξ) and a stabilized legendrian knot Λ, the contact manifold obtained by performing a contact $(+1)$–surgery along Λ is overtwisted. This follows immediately from the Thurston–Bennequin inequality. The statement can also be proven in higher dimensions:

THEOREM 1.2. *Let (Y, ξ) be a contact manifold and $\Lambda \subseteq Y$ a loose Legendrian submanifold. The contact $(+1)$–surgery on (Y, ξ) along Λ is an overtwisted manifold.*

The study of Stein cobordisms has been thouroughly developed [**36**] in higher dimensional contact topology. The existence h–principle and Theorem 1.1 imply the following

THEOREM 1.3. *Let (Y_1, ξ_1) and (Y_2, ξ_2) be two topologically Stein cobordant contact manifolds and (Y_1, ξ_1) overtwisted. Then there exists a Stein cobordism from (Y_1, ξ_1) to (Y_2, ξ_2).*

This result generalizes to higher dimensions the 3–dimensional theorem proven in [**58**]. Theorem 1.3 follows from Theorem 3.1 and [**36**].

1.3. Consequences B. Theorem 1.1 emphasizes the importance of the size of a neighborhood of a contact submanifold. In this direction it is relevant to understand the dichotomy between tight and overtwisted in terms of small and large neighborhoods. This series of consequences only use the equivalence $1 = 2a$.

First, we show that small neighborhoods of any (Y, ξ) give tight contact structures, using a topological assumption on Y.

THEOREM 1.4. *Let $(Y, \ker\alpha)$ be an overtwisted contact manifold with a trivial stable normal bundle. Then $(Y \times D^2(\varepsilon), \ker(\alpha + \lambda_{st}))$ is tight.*

Theorem 1.4 is proven in Section 5. Theorem 1.1 and 1.4 readily imply the following non–squeezing result:

COROLLARY 1.5. *Let $(Y, \ker\alpha)$ be an overtwisted contact manifold with a trivial stable normal bundle. Then there exists $\delta \in \mathbb{R}^+$ such that for* *any $R > \delta$ there is no contact embedding*

$$(Y \times D^2(R), \ker(\alpha + \lambda_{st})) \longrightarrow (Y \times D^2(\delta), \ker(\alpha + \lambda_{st})).$$

Corollary 1.5 partially generalizes and complements [**107**]. The conclusions of Theorem 1.4 and Corollary 1.5 are simpler to achieve if we impose a fillability condition on the contact structure (Y, ξ). The mildest of such hypotheses suffices:

PROPOSITION 1.6. *Let $(Y, \ker \alpha)$ be a weakly fillable contact manifold. Then the contact manifold $(Y \times D^2(\varepsilon), \ker(\alpha + \lambda_{st}))$ is tight.*

In particular Corollary 1.5 also holds under this hypothesis. Proposition 1.6 follows from [**15**] and [**19**]. Thus it is proven without Theorem 1.1 but it is in a sense weaker than Theorem 1.4 because the hypothesis is imposed on the contact structure (Y, ξ) and not only the smooth manifold Y.

In contrast with Corollary 1.5, we can provide a contact squeezing. The h–principle for isocontact embeddings [**15**] into overtwisted manifolds implies the following result:

THEOREM 1.7. *Let $(Y, \ker(\alpha))$ be an overtwisted contact manifold. There exists a radius $R_0 > 0$ such that for any $R > R_0$, there exists a compactly supported contact isotopy f_t of $(Y \times \mathbb{C}, \ker\{\alpha + \lambda_{st}\})$ with $f_1(Y \times D^2(R)) \subseteq Y \times D^2(R_0)$.*

Theorem 1.7 being a contact squeezing result relates to (weak) non–orderability [**15, 53, 72**]. The radius R_0 in the statement of Theorem 1.7 can be taken to be twice the minimal radius R_c such that the contact manifold $(Y \times D^2(R_c), \ker\{\alpha + \lambda_{\mathrm{st}}\})$ is overtwisted.

Regarding the size of neighborhoods, the contact branch cover technique [**107**] along with Theorem 1.1 yield the following class of examples of overtwisted contact structures.

THEOREM 1.8. *Let (Y, ξ) be a contact manifold and $(D, \xi|_D)$ a codimension–2 overtwisted contact submanifold. A k–fold contact branched cover of (Y, ξ) along $(D, \xi|_D)$ is overtwisted for k large enough.*

Theorem 1.8 follows from the equivalence $1 = 2a = 2b$ in Theorem 1.1.

1.4. Consequences B'. The relation between the notions of overtwistedness in different dimensions provided by Theorem 1.1 with the equivalences $1 = 2a = 2b$ have many diverse applications. Let us state three of them.

First, the main theorem in Chapter 6 provides a C^0–bound for the generating Hamiltonian of a positive loop of contactomorphisms in an overtwisted 3–fold. We generalize the result in the following theorem.

THEOREM 1.9. *Let $(Y, \ker\alpha)$ be an overtwisted contact manifold with trivial stable normal bundle. Then there exists a real positive constant $C(\alpha)$ such that any positive loop $\{\phi_\theta\}$ of contactomorphisms which is generated by a contact Hamiltonian $H : Y \times \mathbb{S}^1 \longrightarrow \mathbb{R}^+$ satisfies*

$$\|H\|_{C^0} \geq C(\alpha)\,.$$

Theorem 1.1 is proven in Section 5 as a consequence of Theorem 1.4.

Second, consider a contact manifold $(Y, \ker\alpha)$ and a contactomorphism $f \in \mathrm{Cont}(Y, \ker\alpha)$. For every fixed contact form α, there exists a conformal factor $c_f : Y \longrightarrow \mathbb{R}^+$ defined by $f^*\alpha = c_f\alpha$. The functions c_f are non–vanishing, and we assume they are positive. For instance, given a diffeomorphism $h \in \mathrm{Diff}(\mathbb{R})$, the contactomorphism $f \in \mathrm{Cont}(\mathbb{R}^{2n+1}, \ker\alpha_0)$ defined by

$$(z, r, \theta) \longmapsto f(z, r, \theta) = (h(z), \sqrt{h'(z)}r, \theta)$$

satisfies $c_f(z, r, \theta) = h'(z)$. Choosing $h \in \mathrm{Diff}(\mathbb{R})$ such that $0 < h'(z) < \varepsilon$, $\forall z \in \mathbb{R}$, the example also shows that $\forall\varepsilon > 0$ there exist conformal factors c_f such that $c_f < \varepsilon$.

Note that this is not possible for contact manifolds $(Y, \ker\alpha)$ with finite associated volume $\alpha \wedge d\alpha^n$: in this case the chain rule implies the lower bound $\sup\{c_f\} \geq 1$. Nevertheless, this lower bound holds regardless in the presence of an overtwisted disk:

THEOREM 1.10. *Let $(Y, \ker\alpha)$ be a contact manifold with trivial stable normal bundle. Conisder a contactomorphism $f \in \mathrm{Cont}(Y, \ker\alpha_{ot})$ with a conformal factor $f^*\alpha_{ot} = c_f\alpha_{ot}$. Then $\sup\limits_{p\in Y}\{c_f(p)\} \geq 1$.*

Theorem 1.10 is proven in Section 5.

Third, there are few constructions producing higher dimensional contact manifolds from a given contact manifold [**16, 19**]. The combination of symplectization and contactization being one of the simplest. The behaviour of these two operations is quite diverse as seen in [**37, 52**] and Chapter 8 in this dissertation. The existence h–principle and Theorem 1.1 imply the following result:

THEOREM 1.11. *There exist smooth manifolds* Y *with two non–isomorphic contact structures* $\ker\alpha_1$ *and* $\ker\alpha_2$ *such that* $(Y\times\mathbb{C},\ker\{\alpha_1+\lambda_{st}\})$ *and* $(Y\times\mathbb{C},\ker\{\alpha_2+\lambda_{st}\})$ *are contactomorphic.*

This is an exercise in algebraic topology. For instance, we can consider $\ker\alpha_1$ and $\ker\alpha_2$ to be two different overtwisted contact structures on any 3–fold Y with $H^2(Y,\mathbb{Z})=0$. Then the two hyperplane fields $\ker(\alpha_1+\lambda_{\mathrm{st}})$ and $\ker(\alpha_2+\lambda_{\mathrm{st}})$ become homotopic as almost contact structures in $Y\times\mathbb{C}$.

REMARK 1.12. Note that the symplectizations of two different overtwisted contact structures on $\mathbb{S}^3$ are not symplectomorphic, thus Theorem 1.11 shows that the contactizations of two non–isomorphic exact symplectic structures can be contactomorphic.

These previous two subsections have explored possible applications of $1=2a=2b$. There are many relevant applications of the equivalences $1=3a=3b$ and $1=4$. In order for this Chapter to be as mathematically coherent and self–contained as possible, we will be exploring further consequences of Theorem 1.1 in a future project.

1.5. The argument for Theorem 1.1. Let us detail the logic of the proof for the implications in Theorem 1.1. The existence h–principle in [**15**] readily implies that if (Y,ξ) is overtwisted then $2a$, $2b$, $3a$ and $3b$ hold.

- The equivalence $1=2a=2b$: Since $2a$ implies $2b$, it suffices to show that $2b$ implies 1. This is the content of Theorem 2.1, proved in Section 2.

- The equivalence $1=3a=3b$: This is proven in Section 3 as a consequence of Theorem 3.4. The main ingredient is Lemma 3.1. Note also that $3a$ implies $3b$ follows from [**105**, Theorem 1.1].

- The equivalence $1=4$: This is detailed in Section 4, the argument actually shows that $3a$ is equivalent to 4.

In the method of proof we present in this Chapter, the equivalences $1=3a=3b=4$ strongly use Lemma 3.1 which at the same time relies on $1=2b$. Hence, the order in which we prove the equivalences is relevant: first $1=2b$, second $1=3a=3b$ and third $3a=4$.

1.6. Organization of Chapter 3. There are four sections in the Chapter. The first three contain the argument for Theorem 1.1: Section

2 proves $1 = 2b$, Section 3 shows $1 = 3a = 3b$ and Section 4 concludes $1 = 4$. These sections also contain results that can be of interest on their own. In particular, the connection developed in Section 4 is relevant for higher dimensional contact topology.

1.7. Acknowledgements. Regarding the results of Chapter 3, I am grateful to M.S. Borman and Y. Eliashberg for many useful discussions. I would also like to thank O. van Koert for valuable conversations on the relation between open book decompositions and contact structures.

1.8. Notation. The majority of the notation used along the Chapter is either standard, it appears in Chapter 2 or it is (hopefully) self–explanatory. The equality $(Y, \xi) = ob(W, \lambda, \varphi)$ denotes the fact that the contact structure (Y, ξ) is supported by the open book (W, λ, φ). Since the choice of Liouville form λ is often natural or implied, we also write $(Y, \xi) = ob(W, \varphi)$. In Section 5 we use the notation $D(R_1, R_2, \dots, R_s) = D^2(R_1) \times \dots \times D^2(R_s)$.

2. Thick neighborhoods of overtwisted submanifolds

In this section we begin the proof of Theorem 1.1 with the equivalence $1 = 2$. Since the implications $1 \Rightarrow 2a \Rightarrow 2b$ certainly hold, it suffices to prove $2b \Rightarrow 1$. This is the content of the following theorem.

THEOREM 2.1. *Let $(\mathbb{R}^{2n-3}, \ker \alpha_{ot})$ be an overtwisted contact structure. Then for sufficiently large R, the contact manifold $(\mathbb{R}^{2n-3} \times D^2(R), \ker(\alpha_{ot} + \lambda_{st}))$ is overtwisted.*

Theorem 2.1 and its proof require some preliminaries, including the definition of the overtwisted disk [**15**]. This definition is reviewed in Subsection 2.1. Subsections 2.2 and 2.3 contain technical results for the argument. Then Theorem 2.1 is proven in Subsection 2.4 for the case $n = 2$ and in Subsection 2.6 for $n \geq 3$. This distinction is not essential but it hopefully contributes to a better understanding of the result.

2.1. Overtwisted Disks. Consider cylindrical coordinates

$$(z, u_1, \dots, u_{n-2}, \varphi_1, \dots, \varphi_{n-2}) \in \mathbb{R}^{2n-1} = \mathbb{R} \times (\mathbb{R}^2)^{n-2}$$

with each pair $(\sqrt{u_i}, \varphi_i) \in \mathbb{R}^2$ being polar coordinates. The standard contact structure $(\mathbb{R}^{2n-1}, \xi_0)$ is given by the kernel of the 1–form

$$\alpha_0 = dz + \sum_{i=1}^{n-2} u_i d\varphi_i.$$

In order to ease notation, we denote $u = \sum_{i=1}^{n-2} u_i$. Let $\varepsilon \in \mathbb{R}^+$ be given, define the domains

$\Delta_{\mathrm{cyl}} = \{z \in [-1, 1-\varepsilon],\ u \in [0,1]\}, \quad \Delta_\varepsilon = \{z \in [-1+\varepsilon, 1-\varepsilon],\ u \in [0, 1-\varepsilon]\},$

and consider the subset $B = \{z = -1,\ u \in [0,1]\} \cup \{z \in [-1, 1-\varepsilon],\ u = 1\} \subseteq \partial\Delta_{\mathrm{cyl}}$ of the boundary of Δ_{cyl}. These domains are shown in Figure 1.

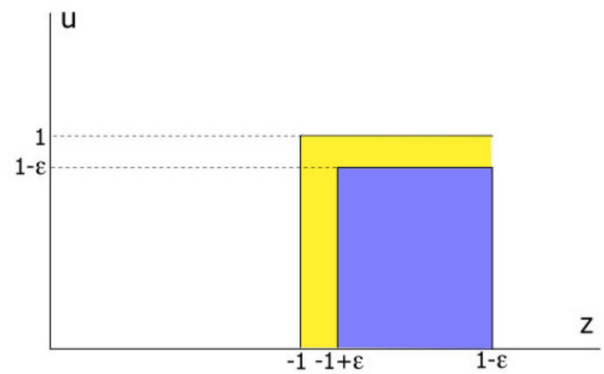


FIGURE 1. The domains Δ in yellow and Δ_ε in blue.

Let $k_\varepsilon : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by

$$k_\varepsilon(x) := \begin{cases} 0 & x \leqslant 1-\varepsilon \\ x-(1-\varepsilon) & x \geqslant 1-\varepsilon. \end{cases}$$

and fix a piecewise smooth function $K_\varepsilon : \Delta_{\mathrm{cyl}} \longrightarrow \mathbb{R}$ of the form

$$K_\varepsilon(u_i, \varphi_i, z) := \begin{cases} k_\varepsilon(z) + k_\varepsilon(u) & (u_i, \varphi_i, z) \in \Delta_{\mathrm{cyl}} \setminus \mathrm{Int}(\Delta_\varepsilon) \\ < 0 & (u_i, \varphi_i, z) \in \mathrm{Int}(\Delta_\varepsilon). \end{cases}$$

Denote $q = (u_i, \varphi_i, z)$ and define two embeddings of hypersurfaces:

$\Sigma_1 = \{(q, v, t) \in \Delta_{\mathrm{cyl}} \times T^*S^1 \ :\ v = K_\varepsilon(q)\} \subseteq (\Delta_{\mathrm{cyl}} \times T^*S^1, \ker(\alpha_0 + v dt))$

$\Sigma_2 = \{(q, v, t) \in \Delta_{\mathrm{cyl}} \times \mathbb{C} \ :\ q \in B,\ v \in [0, K_\varepsilon(q)]\} \subseteq (\Delta_{\mathrm{cyl}} \times \mathbb{C}, \ker(\alpha_0 + v dt)).$

The coordinates $(\sqrt{v}, t)$ represent polar coordinates on $\mathbb{C}$. Notice that $K_\varepsilon > 0$ on $B \subset \partial\Delta_{\mathrm{cyl}}$ and thus Σ_2 is well–defined. Since $\partial\Sigma_2 \subseteq \partial\Sigma_1$, the union $\Sigma_1 \cup \Sigma_2$ is a piecewise smooth disk. Let us denote this disk, together with the germ of contact structure defined by the embedding, by $(D^{\text{ot}}_\varepsilon, \eta^{\text{ot}}_\varepsilon)$.

In the article [**15**] a constant $\varepsilon_{\text{univ}} > 0$ is defined, and it only depends on the dimension. Given any $\varepsilon < \varepsilon_{\text{univ}}$, the contact germ $(D^{\text{ot}}_\varepsilon, \eta^{\text{ot}}_\varepsilon)$ is said to be an *overtwisted disk*. A contact manifold (Y^{2n+1}, ξ) is *overtwisted* if there is an embedded piecewise smooth disk $D^{2n} \subseteq Y$ so that $(D^{2n}, \xi|_{D^{2n}})$ is contactomorphic to an overtwisted disk.

This is the original definition of the overtwisted disk in the first version of [**15**]. Since its appearance, the definition has been modified in order to admit a larger class of functions K_ε. This is the class of special Hamiltonians in [**15**][Section 3.2]. Nevertheless, the uniqueness h–principle [**15**] allows us to use the initial definition given above and prove Theorem 2.1 for this definition.

In the following subsection we define a contact domain used in the argument of Theorem 2.1.

2.2. Contact domains. Let (Y, ξ) be an overtwisted contact 3–fold. In particular, it contains an embedded overtwisted 2–disk with a 1–dimensional domain of definition $\Delta = \Delta(z)$. Theorem 2.1 states the existence of an overtwisted 4–disk in $Y \times \mathbb{D}^2(R)$ for a sufficiently large radius $R \in \mathbb{R}^+$. The 3–dimensional domain $\Delta = \Delta(z, u, \varphi)$ defining this overtwisted 4–disk is contained in $\Delta(z) \times \mathbb{D}^2(R)$. The following discussion is relevant for the proof Theorem 2.1.

In coordinates $(z, u, \varphi) \in \mathbb{R}^3$ consider the abstract disjoint union

$$\widetilde{\Delta} = \{z \in [-1, 1-\varepsilon],\ u \in [0, 1-\frac{\varepsilon}{2}]\} \cup$$

$$\cup \{z \in [-3 + \frac{4}{\varepsilon}(1-u), -1-\varepsilon+\frac{4}{\varepsilon}(1-u)],\ u \in [1-\frac{\varepsilon}{2}, 1]\}.$$

This domain $\widetilde{\Delta}$ is shown in Figure 2. It contains the two subdomains

$$\widetilde{\Delta}_- = \{z \in (-1 + \frac{2\varepsilon}{3}, 1-\varepsilon]\} \subseteq \widetilde{\Delta}$$

$$\widetilde{\Delta}_+ = \{z \in [-3, -1 + \frac{\varepsilon}{3})\} \subseteq \widetilde{\Delta}.$$

Let us consider these as contact domains $(\widetilde{\Delta}, \xi)$ with their induced contact structure as subdomains of $(\mathbb{R}^3, \ker \alpha_0)$. Then we can prove the following

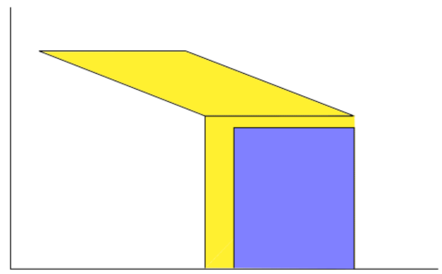


FIGURE 2. The domain $\widetilde{\Delta}$.

LEMMA 2.2. *There is a contactomorphism* $f : (\widetilde{\Delta}, \xi_0) \longrightarrow (\Delta_{cyl}, \xi_0)$ *such that* $\Delta_\varepsilon \subseteq f(\widetilde{\Delta}_-)$ *and* $B \subseteq f(\widetilde{\Delta}_+)$*.*

PROOF. Consider the following contactomorphism in the region $\{u \geq 1 - \varepsilon/2\}$ defined by

$$f(z, u, \varphi) = (z + 4/\varepsilon(u - 1 + \varepsilon/2), u, \varphi - 4/\varepsilon \ln(u)),$$

and extend it to the region $\{u \leq 1 - \varepsilon/2\}$ by

$$(z, u, \varphi) \longmapsto (z, u, \varphi - 4/\varepsilon \ln(1 - \varepsilon/2)).$$

Note that f is well–defined since u is strictly positive on $\{u \geq 1 - \varepsilon/2\}$. This extension defines the required contactomorphism. $\square$

Lemma 2.2 and the 3–dimensional h–principle [**45**] are enough to conclude Theorem 1.1 in the case of a contact 3–fold. The use of the result in [**45**] is controlled and the precise details are provided in Subsection 2.3.

2.3. Local model (M^3, α_M)**.** In the 3–dimensional case, Theorem 2.1 is proven with the use of a local model $(M, \ker \alpha_M)$ which is contained in any overtwisted 3–fold. The domain M is diffeomorphic to an open 3–ball and admits global coordinates (z, v, t). In these coordinates the contact form reads

$$\alpha_M = dz + vdt.$$

Let us describe the domain M in detail. The coordinate $z \in (-3 - \varepsilon, 1)$ dictates the domain of definition of the coordinates (v, t) and the symplectic submanifolds $\{z = \text{constant}\}$ belong to one of the following three types:

a. For $z \in (-1 + \frac{2\varepsilon}{3}, 1)$, we have $(v, t) \in (-\infty, \infty) \times S^1$. Thus in this range the submanifolds $\{z = \text{constant}\}$ are diffeomorphic to T^*S^1 since the restriction of α equals the canonical Liouville form.

b. For $z \in [-1+\frac{\varepsilon}{3}, -1+\frac{2\varepsilon}{3}]$, we let $t \in S^1$ and $v \in (0,\infty)$. Then the fibers are exact symplectomorphic to $\{v > 0\} \subseteq T^*S^1$. Notice that these fibers are also equal to the standard Liouville structure on $\mathbb{C} \setminus \{0\}$ with polar coordinates $(\sqrt{v}, t)$.

c. For $z \in (-3-\varepsilon, -1+\frac{\varepsilon}{3})$, we define the fibers $\{z = \text{constant}\}$ to be equal to $\mathbb{C}$, with $(\sqrt{v}, t)$ continuing to represent polar coordinates.

First, the contact domain $(M, \ker \alpha_M)$ is overtwisted. For example, notice that the Legendrian $\{z = \text{constant} > -1 + \frac{2\varepsilon}{3},\ v = 0\}$ is unknotted and has zero Thurston-Bennequin number.

Second, the contact domain $(M, \ker \alpha_M)$ serves as a local model in any overtwisted 3–fold. Indeed, let (Y, ξ) be an overtwisted 3–fold and choose an open ball $U \subseteq Y$ which is contained in a compact subset. Let us consider a new contact structure ζ on Y, which is homotopic through plane fields to ξ, equal to ξ outside of a compact subset, and equal to $\ker(\alpha_M)$ on $U \cong M$. This can be arranged by the theorem of R. Lutz and J. Martinet [**91, 94**]. Since ζ and ξ are overtwisted and equal outside of a compact subset, the uniqueness h–principle [**45**] implies that they are homotopic with a compactly supported homotopy, and Gray's theorem [**73**] implies that they are isotopic.

Therefore, in the proof of Theorem 2.1 for the case $\dim(Y)=3$, we assume without loss of generality that $(Y, \ker \alpha) = (M, \ker \alpha_M)$.

2.4. Proof of Theorem 2.1 for 3–folds. Let $\varepsilon \in \mathbb{R}^+$ be such that $\varepsilon < \varepsilon_{\text{univ}}$ and consider the contactomorphism $f : (\widetilde{\Delta}, \xi_0) \longrightarrow (\Delta_{\text{cyl}}, \xi_0)$ provided in Lemma 2.2. Consider the function $c_f : \widetilde{\Delta} \longrightarrow (0,\infty)$ defined by the equation $f^*\alpha_{\text{st}} = c_f \alpha_{\text{st}}$ and the Hamiltonian $\widetilde{K} : \widetilde{\Delta} \longrightarrow \mathbb{R}$ given by the equation $c_f \cdot \widetilde{K} = K \circ f$.

We restrict to the local model defined in Section 2.3 and assume that $(Y, \ker \alpha) = (M, \ker \alpha_M)$. If $(\sqrt{u}, \varphi)$ are polar coordinates on $\mathbb{D}^2(1)$, then (z, v, t, u, φ) define coordinates on $M \times \mathbb{D}^2(1)$ and this domain is given the contact form $dz + vdt + ud\varphi$.

Let us detail the overtwisted 4–disk in $M \times \mathbb{D}^2(1)$. Consider the two hypersurfaces

$$\widetilde{\Sigma}_1 = \{(z, v, t, u, \varphi) \,:\, v = \widetilde{K}(z, u, \varphi)\}$$
$$\widetilde{\Sigma}_2 = \{(z, v, t, u, \varphi) \,:\, f(z, u, \varphi) \in B, \; v \in [0, \widetilde{K}(z, u, \varphi)]\}.$$

Notice that $\widetilde{\Sigma}_1$ is a well–defined subset of $M \times D^2(1)$ since $\Delta_\varepsilon \subseteq f(\widetilde{\Delta}_-)$, and $\widetilde{\Sigma}_2$ is also well–defined since $B \subseteq f(\widetilde{\Delta}_+)$. Then the contact germ of the 4–disk $D^4 = \Sigma_1 \cup \Sigma_2$ an overtwisted disk. Indeed, there exists a contactomorphism taking D^4 to the standard model D^{ot} given by the stabilization of f. That is, the contactomorphism

$$F(z, u, \varphi, v, t) = (f(z, u, \varphi), v \cdot S_f(z, u, \varphi), t).$$

maps the contac germ $(D^4, \ker(\alpha_M + \lambda_{\text{st}}))$ to the contact germ $(D^{\text{ot}}, \eta^{\text{ot}})$ and thus the contact domain $(M \times \mathbb{D}^2(1), \ker(\alpha + \lambda_{\text{st}}))$ contains an overtwisted 4–disk. $\square$

2.5. Local model (M, α_M)**.** The argument used in order to conclude Theorem 2.1 for an arbitrary overtwisted (Y, ξ) contains the same steps as in the case $n = 2$. However, the definition of the domain $\widetilde{\Delta}$ and the local model (M, α_M) are more involved.

Instead of directly defining the contact domain $(\widetilde{\Delta}, \xi_0)$ and then provide an explicit contactomorphism to Δ with the necessary properties, we start with (M, α_M) and then construct $(\widetilde{\Delta}, \xi_0)$ with a contact flow defined on $(\Delta, \xi_0) \subseteq (\mathbb{R}^{2n-1}, \xi_0)$.

In the case $\dim(Y)$=3 we have local coordinates (z, v, t) in a neighborhood of the overtwisted 2–disk and the local model (M, α_M) is defined in terms of (z, u_0, φ_0), where the coordinates (u_0, φ_0) belong to $\mathbb{D}^2(R)$. In higher dimension $\dim(Y)= 2n-1$, the local coordinates in a neighborhood of an overtwisted $(2n-2)$–disk D_{ot} are (z, u, φ, v, t) and the local model (M, α_M) has coordinates $(z, u, \varphi, u_0, \varphi_0)$.

In the previous case the domain of the variables (v, t) depended on z, now the dependence is on the coordinates $(z, u, \varphi) \in \Pi^{2n-2}(\rho)$, where Π is a polydisk and $\rho \in \mathbb{R}^+$ is fixed and large enough. The domain of the variables (v, t) is still either T^*S^1, $\mathbb{C}^*$ or $\mathbb{C}$. Let us consider global coordinates (z, u, φ, v, t) and the contact form $dz + ud\varphi + vdt$. The coordinates (z, u, φ) dictate the domain of definition of (v, t) as follows:

a. For $\{(z, u) \in (-1 + \frac{2\varepsilon}{3}, \rho) \times [0, 1 - \frac{2\varepsilon}{3})\}$, we have $(v, t) \in (-\infty, \infty) \times S^1$. In this range, the symplectic submanifolds $\{(z, u, p) = \text{constant}\}$ are symplectomorphic to T^*S^1.

b. For $\{(z,u) \in (-1+\frac{1\varepsilon}{3}, -1+\frac{2\varepsilon}{3}) \times [1-\frac{2\varepsilon}{3}, 1-\frac{\varepsilon}{3})\}$, we consider $(v,t) \in (0,\infty) \times S^1$. The symplectic submanifolds $\{(z,u,p) = \text{constant}\}$ are symplectomorphic to $\mathbb{C}^*$.

c. For $\{(z,u) \in (-\rho, -1+\frac{1\varepsilon}{3}) \times [1-\frac{\varepsilon}{3}, \rho)\}$, we have $(v,t) \in \mathbb{C}$.

This is the local model $(M, \ker\alpha_M)$ inserted in the overtwisted contact manifold $(Y, \ker\alpha)$ using the uniqueness h–principle [**15**]. Hence in order to conclude Theorem 2.1 it is left to prove that the contact domain $(M \times \mathbb{D}^2(R), \ker(\alpha_M + \lambda_{\mathrm{st}}))$ contains an overtwisted $2n$–disk.

2.6. Proof of Theorem 2.1. The germ of an overtwisted $(2n-2)$–disk in (Y,ξ) has local coordinates $(z,u,\varphi; v,t)$ and its domain is the $(2n-3)$–dimensional subset $\Delta = \Delta(z,u,\varphi)$.

In the 3–dimensional case we provided a domain $\widetilde{\Delta}$ and an explicit contactomorphism

$$f : (\widetilde{\Delta}, \xi_0) \longrightarrow (\Delta, \xi_0)$$

such that $\Delta_\varepsilon \subseteq f(\widetilde{\Delta}_-)$ and $B \subseteq f(\widetilde{\Delta}_+)$. Instead, let us begin with the contact domain

$$\Delta = \Delta(z,u,\varphi; u_0, \varphi_0) \subset (\Delta(z,u,\varphi) \times \mathbb{D}^2(u_0,\varphi_0), \ker\lambda_{st})$$

and construct the contact domain $(\widetilde{\Delta}, \xi_0)$.

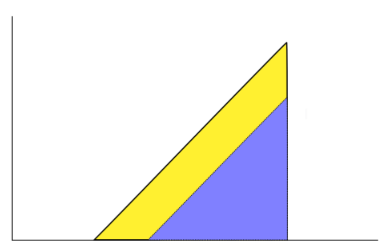


FIGURE 3. Cross section (u, u_0) of the domain $\Delta = (z,u,\varphi,u_0,\varphi_0)$ at $z = C$ for a constant $C \in (-1+\varepsilon, 1-\varepsilon)$.

Consider the contact vector field $X = \partial_u + \partial_{u_0} + 2z\partial_z$ on (Δ, ξ_0) and cut–off its contact Hamiltonian H to a Hamiltonian $\widetilde{H}$ such that its contact vector field $\widetilde{X}$ satisfies

a. $\widetilde{X}$ vanishes in $\{z \geq -1+2\varepsilon/3, u+u_0 \leq 1-2\varepsilon/3\}$.
b. $\widetilde{X}$ coincides with X in $\{z \leq -1+\varepsilon/3, 1-\varepsilon/3 \leq u+u_0 \leq 1\}$.

The flow of this vector field $\widetilde{X}$ expands the domain (Δ, ξ_0). For a large enough τ, the image $\varphi^\tau_{\widetilde{X}}(\Delta)$ contains a small region in which the coordinates (v,t) belong to T^*S^1. See Figures 3 and 4. This region is a small

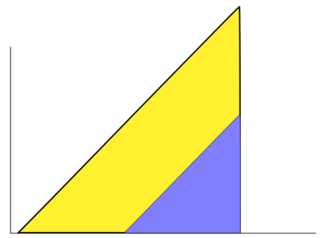


FIGURE 4. Cross section (u, u_0) for the expanded domain $\varphi^{\tau_0}_{\widetilde{X}}(\Delta)$ at $z = C$ for a constant $C \in (-1+\varepsilon, 1-\varepsilon)$. We need to flow for a time $\tau > \tau_0$.

neighborhood of the area $\{u_0 = 1, u = 0\}$ and is the only part left to fill. Let τ be large enough such that this area is triangular in a cross section (u, u_0). In order to push this area to a region where $(v, t) \in \mathbb{C}$ we apply the contactomorphism provided by Lemma 2.2 in the 3–dimensional domain (z, u_0, φ_0). This image domain is the analogue of $\widetilde{\Delta}$ in higher dimensions.

The local model (M, α_M) has been defined, and it can be inserted in any overtwisted manifold. Since we have obtained $\widetilde{\Delta}$, the same argument than in the 3–dimensional case concludes Theorem 2.1. □

Theorem 2.1 implies the first equivalence in Theorem 1.1. The following section establishes the equivalence relating looseness of the Legendrian unknot and overtwistedness.

3. Weinstein cobordism from overtwisted to standard sphere

The main goal of this section is proving the equivalences $1 = 3a = 3b$ in Theorem 1.1. These equivalences are proven using the following two theorems:

THEOREM 3.1. *In every dimension, there is a Weinstein cobordism* (W, λ, φ) *such that the concave end* $(\partial_- W, \lambda)$ *is overtwisted and the convex end* $(\partial_+ W, \lambda) \cong (\mathbb{S}^{2n-1}, \xi_0)$.

Theorem 3.1 is proven assuming the equivalence $1 = 2a$ in Theorem 1.1 which has been proven in Section 2. Then Theorem 3.1 is used in the proof of the rest of equivalences $1 = 3a = 3b = 4$.

THEOREM 3.2. *Let* (Y, ξ) *be a contact manifold with an open proper subset* $U \subseteq Y$ *such that for any contact manifold containing (a contactomorphic copy of)* U*, every Legendrian on the complement is loose. Then the contact manifold* (Y, ξ) *is overtwisted.*

Theorem 3.2 implies the following result.

THEOREM 3.3. *Let Λ_0 be the standard Legendrian unknot inside a contact manifold (Y, ξ). If Λ_0 is a loose Legendrian then (Y, ξ) is overtwisted.*

Similarly, Theorem 3.2 and [**105**, Theorem 1.1] lead to

THEOREM 3.4. *Let (Y, ξ) be a contact manifold containing a small plastikstufe with spherical core and trivial rotation. Then (Y, ξ) is overtwisted.*

3.1. Proof of Theorem 3.1. We construct a Weinstein cobordism $(W^{2n}, \lambda, \varphi)$ of finite type from an overtwisted contact structure $(\mathbb{S}^{2n-1}, \xi_{ot})$ to the standard contact sphere $(\mathbb{S}^{2n-1}, \xi_0)$. The construction has two steps.

First, we prove that the contact manifold $(\mathbb{S}^{2n-1}, \xi_k) = ob(A_{2k-1}, \tau_1^{-1} \circ \ldots \circ \tau_{2k-1}^{-1})$ is overtwisted for k large enough. Second, there exists a Weinstein cobordism (W, λ, φ)

$$(\mathbb{S}^{2n-1}, \xi_k) = ob(A_{2k-1}, \tau_1^{-1} \circ \ldots \circ \tau_{2k-1}^{-1}) \stackrel{W}{\Longrightarrow} (\mathbb{S}^{2n-1}, \xi_0) = ob(A_{2k-1}, \tau_1 \circ \ldots \circ \tau_{2k-1}).$$

Then Theorem 3.1 follows by choosing $(\mathbb{S}^{2n-1}, \xi_{ot}) = (\mathbb{S}^{2n-1}, \xi_k)$ for k large enough. Here A_k denotes the A_k Milnor fibre obtained as an A_k–plumbing of k cotangent bundles of spheres with its induced Weinstein structure.

<u>Assertion 1</u>: $(\mathbb{S}^{2n-1}, \xi_k)$ is overtwisted for k large enough.

By the inductive character of the argument, it suffices to show that $(\mathbb{S}^5, \xi_k)$ is overtwisted for k large enough. The contact manifold $(\mathbb{S}^3, \xi_1) = ob(A_1, \tau^{-1})$ is overtwisted because the zero section of the page is an unknot with $tb = 1$. It also admits a contact embedding into $(\mathbb{S}^5, \xi_1) = ob(A_1, \tau^{-1})$ compatible with the open book decompositions which corresponds to an unknotted equatorial $\mathbb{S}^3 \subseteq \mathbb{S}^5$. Then Theorem 1.8 implies that the k–branched cover (Y_k, ζ_k) of $(\mathbb{S}^5, \xi_1)$ along $(\mathbb{S}^3, \xi_1)$ is an overtwisted contact manifold. Note that Y_k is diffeomorphic to $\mathbb{S}^5$ because $\mathbb{S}^3$ is a smooth unknot.

Let us show that the contact structure $(\mathbb{S}^5, \zeta_k)$ is supported by the open book $ob(A_{2k-1}, \tau_1^{-1} \circ \ldots \circ \tau_{2k-1}^{-1})$ and hence it is contact isotopic to $(\mathbb{S}^5, \xi_k)$, which concludes the assertion. In order to show that the open book $ob(A_{2k-1}, \tau_1^{-1} \circ \ldots \circ \tau_{2k-1}^{-1})$ supports $(\mathbb{S}^5, \zeta_k)$ we can argue as follows. First note that the projection map for the open book $ob(A_1, \tau_1^{-1})$ can be assumed to be the argument of the map

$$f : \mathbb{S}^5 \subset \mathbb{C}^3 \longrightarrow \mathbb{C}, \quad f(z_1, z_2, z_3) = \overline{z}_1^2 + \overline{z}_2^2 + \overline{z}_3^2.$$

Then the overtwisted submanifold $(\mathbb{S}^3, \xi_1)$ is cut out by the equation $\{z_1 = 0\}$ and the k–branched cover along it can be realized by the map $z_1 \longmapsto z_1^k$. Thus the contact structure (Y_k, ζ_k) is supported by the open book induced by the argument of the map

$$f : \mathbb{S}^5 \subset \mathbb{C}^3 \longrightarrow \mathbb{C}, \quad f(z_1, z_2, z_3) = \overline{z}_1^{2k} + \overline{z}_2^2 + \overline{z}_3^2,$$

which is $ob(A_{2k-1}, \tau_1^{-1} \circ \ldots \circ \tau_{2k-1}^{-1})$. □

Assertion 2: $(\mathbb{S}^{2n-1}, \xi_k)$ is cobordant to $(\mathbb{S}^{2n-1}, \xi_0)$.

Suppose $(Y, \xi) = ob(X, \phi)$ is a contact manifold and $\Lambda \subseteq (Y, \xi)$ is a Legendrian sphere isotopic to a Lagrangian sphere $L \subset X$. Then a critical handle attachment along Λ induces a Weinstein cobordism from the original $(Y, \xi) = ob(X, \phi)$ to the surgered contact manifold $(Y_\Lambda(-1), \xi_\Lambda(-1)) = ob(X, \tau_L \circ \phi)$ [**87**]. This general fact can be used in the situation above.

Then there exists a Weinstein cobordism (W, λ, φ)

$$(\mathbb{S}^{2n-1}, \xi_k) = ob(A_k, \tau_1^{-1} \circ \ldots \circ \tau_k^{-1}) \xrightarrow{W} (\mathbb{S}^{2n-1}, \xi_0) = ob(A_k, \tau_1 \circ \ldots \circ \tau_k).$$

obtained by $2k$ critical handle attachments, 2 along each of the k zero sections of the cotangent bundles conforming the A_k configuration. □

3.2. Proof of Theorem 3.2. In order to prove Theorem 3.2 we use Theorem 3.1 and the h–principle for loose Legendrian embeddings which we now state for completeness:

THEOREM 3.5 ([**104**]). *Let Λ_0 be a loose Legendrian, and let $f_t : \Lambda \to Y$ be a smooth isotopy so that f_0 is the inclusion map, and f_1 is unconstrained other than being a smooth embedding. Then there is a Legendrian isotopy $g_t : \Lambda \to Y$ which is C^0–close to f_t.*

Suppose that (Y, ξ) is a contact submanifold with an open proper subset $U \subseteq Y$ such that for any contact manifold (Y', ξ') containing U, every Legendrian on the complement $Y' \setminus U$ is loose. Consider the Weinsten cobordism $(\widetilde{W}, \widetilde{\lambda}, \widetilde{\varphi})$ obtained by contact connect summing Y onto every level set of the Weinstein cobordism W provided by Theorem 3.1. We perform these sums away from all the descending manifolds in (W, λ, φ) and away from $U \subseteq Y$. In particular, $(\widetilde{W}, \widetilde{\lambda}, \widetilde{\varphi})$ is a Weinstein cobordism from an overtwisted contact structure $(Y, \xi_{ot}) = \partial_- W \#(Y, \xi)$ to the original contact structure (Y, ξ).

Break $(\widetilde{W}, \widetilde{\lambda}, \widetilde{\varphi})$ into elementary cobordisms

$$\widetilde{W} = ((-\infty, c_1] \times \partial_- W \#(Y, \xi)) \cup \widetilde{\varphi}^{-1}([c_1, c_2]) \cup \ldots \cup \widetilde{\varphi}^{-1}([c_{k-1}, c_k]) \cup [c_k, \infty) \times (Y, \xi).$$

Then all handle attaching spheres $\Lambda_j \subseteq \widetilde{\varphi}^{-1}(c_j)$ are either subcritical or loose Legendrians. In the language of [**36**], $(\widetilde{W}, \widetilde{\lambda}, \widetilde{\varphi})$ is a flexible Weinstein cobordism.

We show by induction that each contact manifold $\widetilde{\varphi}^{-1}(c_j)$ is overtwisted. The $j = 0$ case follows from the fact that $\partial_- W$ is overtwisted, and the $j = k$ case implies the result. The contact manifold $\widetilde{\varphi}^{-1}(c_{j+1})$ is obtained from $\widetilde{\varphi}^{-1}(c_j)$ by a single Weinstein surgery along Λ_j. Any smooth isotopy of Λ_j can be C^0–approximated by a contact isotopy. Indeed, if Λ_j is subcritical this follows from the h-principle for subcritical isotropic submanifolds [**74**], and if Λ_j is a loose Legendrian this is Theorem 3.5. In particular, we can find a contact isotopy which makes Λ_j disjoint from any overtwisted disk in $\widetilde{\varphi}^{-1}(c_j)$. □

4. Stabilization of Legendrians and open books

In this section we prove the equivalence $3a = 4$ in Theorem 1.1. In fact, the section relates two known procedures in contact topology: the stabilization of a Legendrian submanifold and the (negative) stabilization of a compatible open book. This is explained in Subsection 4.3. The link between these two procedures can be established through Lagrangian surgery [**111**], also referred to as Polterovich surgery. The details regarding Lagrangian surgery are detailed in Subsection 4.2.

The results in Subsections 4.2 and 4.3 imply the following result.

Theorem 4.1. *Let (S^{2n-1}, ξ_-) be the contact manifold defined by an open book whose page is T^*S^{n-1} and whose monodromy is the left handed Dehn twist. Then the standard Legendrian unknot in (S^{2n-1}, ξ_-) is loose.*

This theorem is the essential ingredient for $3a = 4$.

4.1. Legendrians in open books.

Let us start with a theorem, due to E. Giroux:

Theorem 4.2 (E. Giroux). *Let $(Y, \xi) = ob(W, \varphi)$ be a contact manifold and consider the contact structure $(S^{2n-1}, \xi_-) = ob(T^*S^{n-1}, \tau^{-1})$. Then any negative stabilization of the open book (W, φ) is adapted to the contact structure $(Y \# S^{2n-1}, \xi \# \xi_-)$.*

The fact that any overtwisted contact manifold admits a negatively stabilized open book follows quickly from known results. Indeed, let (Y, ξ) be an overtwisted contact structure. Since the set of almost contact structures on the sphere forms a group, the existence theorem from [**15**] implies that there is an overtwisted contact structure (Y, η) so that $(Y \# S^{2n-1}, \eta \# \xi_-)$ is in the same homotopy class of almost contact structures as (Y, ξ). Since the contact structures ξ and $\eta \# \xi_-$ are both overtwisted, they are necessarily isotopic. By E. Giroux's existence theorem for open books compatible with a given contact structure [**71**], the contact structure (Y, η) is compatible with an open book (W, φ). The negative stabilization of (W, φ) is adapted to $(Y, \eta \# \xi_-)$, which is isotopic to (Y, ξ). This shows that $1 = 3a$ implies 4.

In order to prove Theorem 4.1, we develop some combinatorics for describing Legendrians in open books. Let (W, λ, φ) be an abstract open book compactible with (Y, ξ), composed of a Liouville manifold (W, λ) and φ a compactly supported exact symplectomorphism of W. Given an exact Lagrangian $L \subseteq (W, \lambda)$, we can lift it to a Legendrian $\Lambda \subseteq (Y, \xi)$ by letting θ be the path integral of λ, where θ is the coordinate coming from open book coordinates $\xi = \ker(d\theta - \lambda)$. Note that if the Lagrangian L is connected then the path integral is only defined up to a shift by a constant, but it is well defined up to isotopy since constant shifts in θ induce the Reeb flow. If L has multiple components which are allowed to intersect each other, the isotopy type of the Legendrian link depends on this choice of constant.

Given a contact manifold (Y, ξ) the Giroux correspondence provides an open book $(Y, \xi) = ob(W, \lambda, \varphi)$. However, the (Y, ξ) is uniquely determined by φ up to conjugation by compactly supported symplectomorphisms, and in the subsequent discussion this conjugation must also be applied to Lagrangian submanifolds. If $L \subseteq (W, \lambda)$ is a Lagrangian, the Legendrian defined by (W, λ, φ, L) is isotopic to the Legendrian defined by $(W, \lambda, \psi \circ \varphi \circ \psi^{-1}, \psi(L))$, and typically distinct from the Legendrian defined by $(W, \lambda, \psi \circ \varphi \circ \psi^{-1}, L)$. Note also that the Legendrian (W, λ, φ, L) is isotopic to $(W, \lambda, \varphi, \varphi(L))$, since the Reeb flow from time 0 to 2π gives an isotopy between them. These observations are relevant to the proof of Theorem 4.1.

The next subsection contains the results expressing Lagrangian surgery on two Lagrangians (which for us lie on the page of an open book) in terms of the Legendrian connected sum of their Legendrian lifts.

4.2. Lagrangian Surgery and Legendrian Sums. Compactly supported exact symplectomorphisms of a Liouville domain are often times given as compositions of Dehn–Seidel twists [**117**][Chapter I.2]. It is thus relevant to reinterpret the action of Dehn twists on Lagrangians in terms of their Legendrian lifts. This is the aim of this subsection.

We focus on the case where $L \subseteq (W, \lambda)$ is an exact Lagrangian and $S \subseteq W$ is a Lagrangian sphere intersecting L in one point. In this case, the Dehn twist of L around S can be interpreted as the the Polterovich surgery [**60, 111**] of L and S, denoted by $L + S$:

Theorem 4.3 ([**118**]). *The Lagrangian surgery $L + S$ is Lagrangian isotopic to $\tau_S^{-1}(L)$.*
The Lagrangian surgery $S + L$ is Lagrangian isotopic to $\tau_S(L)$.

We now model this operation in terms of the fronts of Legendrian lifts Λ and Σ of the Lagrangians L and S. The conclusion can be stated as follows:

Theorem 4.4. *The Legendrian lift of $L+S$ is isotopic to the Legendrian cusp sum of Λ and Σ. The Legendrian lift of $S + L$ is isotopic to the Legendrian cone sum of Λ and Σ.*

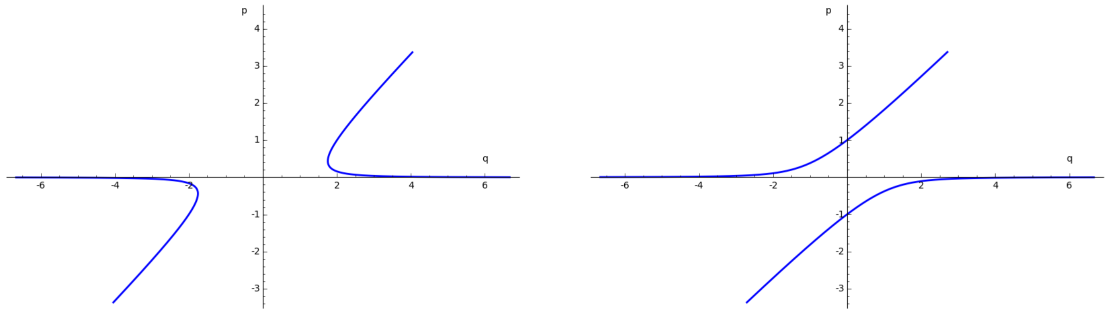
The rest of the subsection proves Theorem 4.4.

Consider local coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n) \in \mathbb{R}^{2n}$ such that $L = \{p_1 = 0, \ldots, p_n = 0\}$, $S = \{q_1 = p_1, \ldots, q_n = p_n\}$ and the Liouville form reads

$$\lambda = \sum_{i=1}^{n} p_i dq_i.$$

In the contactization $(\mathbb{R}^{2n+1}(q, p; z), \ker(dz - \lambda))$ of the exact symplectic manifold $(\mathbb{R}^{2n}(q, p), \lambda)$, the Lagrangian L lifts to the Legendrian $\Lambda = \{(q_1, \ldots, q_n, 0, \ldots, 0; 0)\}$ and the Lagrangian S lifts to the Legendrian $\Sigma = \{(q_1, \ldots, q_n, 0, \ldots, 0; (q_1^2 + \ldots + q_n^2)/2)\}$.

The Lagrangian surgeries $L + S$ and $S + L$ are respectively described in terms of the Lagrangian handles $\Gamma^{\pm}$. In order to parametrize them we use coordinates $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$. These Lagrangian handles are depicted in Figure 5.



(A) The Lagrangian handle Γ^+. (B) The Lagrangian handle Γ^-.

FIGURE 5. The Lagrangian handles $\Gamma^\pm \subseteq \mathbb{R}^{2n}(q,p)$.

First, we consider the case of the Lagrangian handle Γ^+. Let us describe it via the parametrization $\Gamma^+ : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^{2n}$ defined as

$$\Gamma^+(t_1,\dots,t_n) = \big((\mu+\mu^{-1})t_1,\dots,(\mu+\mu^{-1})t_1, \mu t_1,\dots,\mu t_n\big) \text{ where } \mu = \sum_{i=1}^n t_i^2.$$

Note that we have $\lim_{\mu\to\infty} \Gamma^+ \subseteq S$ and $\lim_{\mu\to 0} \Gamma^+ \subseteq L$. We can lift the exact Lagrangian Γ^+ to the contactization via $z = z(t_1,\dots,t_n)$:

$$dz(t) = \sum_{i=1}^n \mu t_i d\left((\mu+\mu^{-1})t_i\right) = \sum_{i=1}^n (\mu^2+1)t_i dt_i + \sum_{i=1}^n \mu t_i^2(1-\mu^{-2})d\mu =$$

$$= \sum_{i=1}^n (\mu^2+1)t_i dt_i + (\mu^2-1)d\mu$$

Hence the partial derivatives of $z(t)$ are:

$$\partial_i z(t) = (\mu^2+1)t_i dt_i + (\mu^2-1)2t_i dt_i = (3\mu^2-1)t_i dt_i.$$

Thus the z–coordinate of the lift is parametrized by $z(t) = \frac{1}{2}(\mu^3-\mu)$ and in the front projection $\mathbb{R}^{n+1}(q_1,\dots,q_n,z)$ we obtain a rotationally symmetric cusp. Part of the front projections in dimensions 3 and 5 are depicted in Figures 6 and 7. This describes the Polterovich surgery $L+S$ in terms of the cusp-sum of the two Legendrians Λ and Σ respectively lifting L and S. This concludes the first statement of Theorem 4.4.

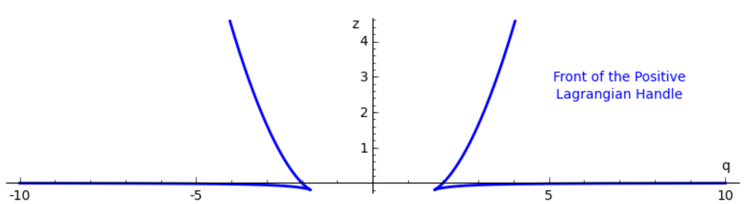


FIGURE 6. Front projection to $\mathbb{R}^2(q_1,z)$ of the Legendrian lift of the positive Lagrangian handle $\Gamma^+ \subseteq \mathbb{R}^3(q_1,p_1,z)$ for $t \in [-1.5,-0.1] \cup [0.1,1.5]$.

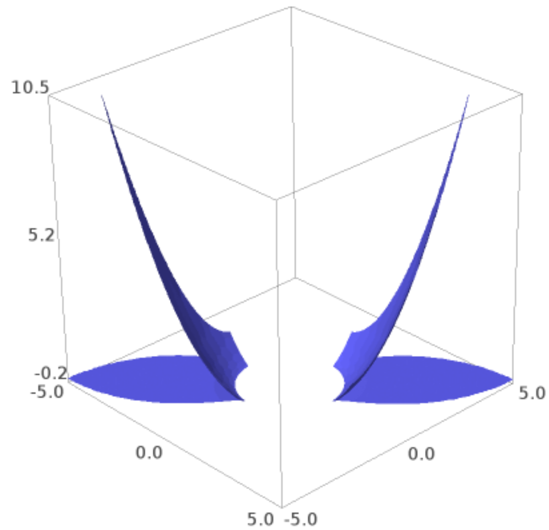


FIGURE 7. Front projection to $\mathbb{R}^3(q_1, q_2, z)$ of the Legendrian lift of $\Gamma^+ \subseteq \mathbb{R}^5$ with parameters (t_1, t_2) in the range $[-1.2, -0.1] \times [-1.2, -0.1] \cup [0.1, 1.2] \times [0.1, 1.2]$.

The Polterovich surgery $S + L$ is described in terms of the Lagrangian handle Γ^-, which yields the cone-sum. Indeed, consider the parametrization of the handle

$$\Gamma^- : \mathbb{R}^n \backslash \{0\} \longrightarrow \mathbb{R}^{2n}, \quad \Gamma^-(t_1, \ldots, t_n) = \left((\mu - \mu^{-1})t_1, \ldots, (\mu - \mu^{-1})t_1, \mu t_1, \ldots, \mu t_n\right).$$

We also have $\lim\limits_{\mu\to\infty} \Gamma^- \subseteq S$ and $\lim\limits_{\mu\to 0} \Gamma^- \subseteq L$. The z–coordinate of the lift to the contactization satisfies

$$dz(t) = \sum_{i=1}^{n} \mu t_i d\left((\mu - \mu^{-1})t_i\right) = \sum_{i=1}^{n} (\mu^2 - 1)t_i dt_i + (\mu^2 + 1)d\mu.$$

We conclude that the partial derivatives of $z(t)$ are $\partial_i z(t) = (3\mu^2 + 1)t_i dt_i$ and $z(t) = \frac{1}{2}(\mu^3 + \mu)$ provides a lift for Γ^-. These front projections are depicted in Figures 8 and 9. This concludes the second statement of Theorem 4.4.

The description provided by Theorem 4.4 is used to prove Theorem 4.1.

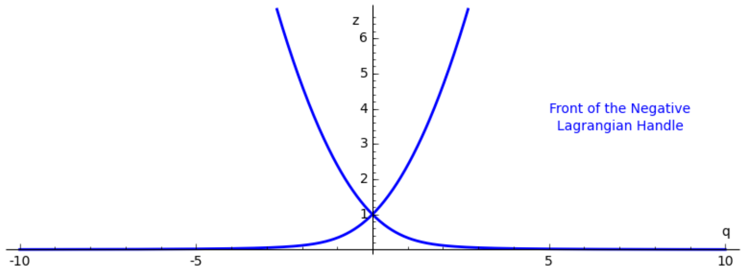


FIGURE 8. Front projection to $\mathbb{R}^2(q_1, z)$ of the Legendrian lift of the handle $\Gamma^- \subseteq \mathbb{R}^3(q_1, p_1, z)$ with $t \in [-1.5, -0.1] \cup [0.1, 1.5]$.

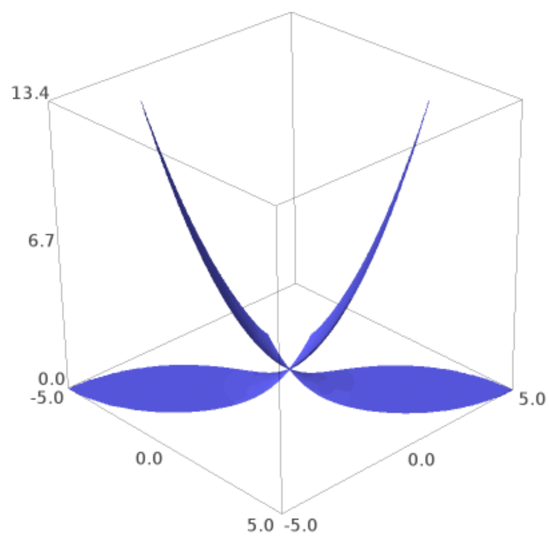


FIGURE 9. Front projection to $\mathbb{R}^3(q_1, q_2, z)$ of the Legendrian lift of $\Gamma^- \subseteq \mathbb{R}^5$ with parameters $(t_1, t_2) \in [-1.2, -0.1] \times [-1.2, -0.1] \cup [0.1, 1.2] \times [0.1, 1.2]$.

4.3. Loose knots in open books. In order to show that the Legendrian unknot in the contact manifold $(\mathbb{S}^{2n-1}, \xi) = ob(A_1, \tau^{-1})$ is loose, we need an understanding of looseness and the standard unknot in the open book framework. This is the content of Propositions 4.5 and 4.6.

PROPOSITION 4.5. *Let $(Y, \xi) = ob(W, \lambda, \varphi)$ be a contact manifold and $(W \cup H, \lambda, \varphi \circ \tau_S)$ a positive stabilization, where $S \subseteq W \cup H$ is the Lagrangian sphere given as the union of the stabilizing disk and the core of H. The Legendrian lift of S to (Y, ξ) is the standard unknot.*

PROOF. First, we note that positive stabilization of an open book can be thought of as connect summing (Y, ξ) with $(S^{2n-1}, \xi_0) = ob(T^*S^{n-1}, \lambda_{\mathrm{st}}, \tau_S)$, where S denotes the zero section. Therefore it suffices to show that the Legendrian lift of S in this one model is the standard unknot. For this, notice that this open book can be thought of as the boundary of the Lefschetz fibration $f : \mathbb{C}^n \to \mathbb{C}$ $f(z_1, \dots, z_n) = z_1^2 + \dots z_n^2$. Then S is Hamiltonian isotopic to a vanishing cycle of the unique critical point at 0. Since $\lambda_{\mathrm{st}}|_S = 0$, the Legendrian lift of S is simply the inclusion of S into a single page of the open book, meaning that f is constant of S.

Taking any path in $\gamma : [0,1] \to \mathbb{C}$ satisfying $\gamma(0) = 0$ and $\gamma(1) = f(S)$, we can use symplectic parallel transport to find a Lagrangian disk $L_\gamma \subseteq \mathbb{C}^n$ so that $\partial L_\gamma = S$. One definition of the standard Legendrian unknot is the Legendrian which is the boundary of the standard Lagrangian plane $\Lambda_0 = \{y_i = 0\} \cap S^{2n-1}_{\mathrm{st}} \subseteq \mathbb{C}^n$. Therefore, it suffices to show that L_γ is Hamiltonian isotopic to the flat Lagrangian plane.

Since $L_\gamma \cap f^{-1}(\gamma(t))$ is Hamiltonian isotopic to the zero section S, we can choose coordinates so that $\lambda|_{L_\gamma} = 0$. This means that the flow of the Liouville vector field, which is a conformal symplectomorphism, takes L_γ to a submanifold which is C^1–close to its linearization at $0 \in \mathbb{C}^n$. Using Moser's theorem, we see that L_γ is Hamiltonian isotopic to a flat plane. $\square$

PROPOSITION 4.6. *Let $(W \cup H, \lambda, \varphi \circ \tau_S)$ be a positively stabilized open book and $L \subseteq W \cup H$ be an exact Lagrangian which transversely intersects S in one point. Then the Legendrian $(W \cup H, \lambda, \varphi \circ \tau_S, L)$ is isotopic to the Legendrian $(W \cup H, \lambda, \varphi \circ \tau_S, \tau_S^{-1}(L))$ and the Legendrian $(W \cup H, \lambda, \varphi \circ \tau_S, \tau_S(L))$ is loose.*

PROOF. Choose a Legendrian lift for L which has $\theta = 0$ at $L \cap S$, and a Legendrian lift for S which is just $\theta = \varepsilon$ for some small constant $\varepsilon > 0$ (this is a Legendrian lift since $\lambda|_S = 0$). Theorem 4.4 implies that the Legendrian lifts of $\tau_S(L)$ and $\tau_S^{-1}(L)$ are the cone and cusp sums of the corresponding Lagrangians. Indeed, since they intersect in one point, we know by Theorem 4.3 that $\tau_S(L) = L + S$ and $\tau_S^{-1} L = S + L$. The Legendrian lift of $L+S$ corresponds to the cusp-sum, and the Legendrian lift of $S + L$ corresponds to their cone-sum. Since S is the Legendrian unknot it is contained in a Darboux ball which is disjoint from L, and since any two Darboux balls are contact isotopic we see that cone or cusp summing with the unknot is a local operation of L.

For the cone sum $(W \cup H, \lambda, \varphi \circ \tau_S, \tau_S^{-1}(L))$, we note that cone-summing a Legendrian with a small Legendrian unknot does not change the Legendrian isotopy type since this is just the $\mathbb{S}^{n-2}$–spinning of the standard Riedemeister I move. Therefore the Legendrian lift of L is isotopic to the lift of $S + L = \tau_S^{-1}(L)$.

For the cusp sum $(W \cup H, \lambda, \varphi \circ \tau_S, \tau_S(L))$, observe that the cusp-sum of a Legendrian with a small Legendrian unknot explicitly creates a loose chart [**36, 104**] and therefore $\tau_S(L) = L + S$ is loose. $\square$

4.4. Proof of Theorem 4.1. Consider the contact manifold

$$(\mathbb{S}^{2n-1}, \xi) = ob(A_1, \tau_L^{-1})$$

and stabilize the open book using a cotangent fiber. The Weinstein page $(W, \lambda) = T^*S^{n-1} \cup H$ of the resulting open book is a plumbing of two copies of T^*S^{n-1}, and note that their zero sections by L and S intersect in one point.

The Legendrian $(W, \lambda, \tau_L^{-1} \circ \tau_S, \tau_S(L))$ is loose by Lemma 4.6 and the Legendrian $(W, \lambda, \tau_L^{-1} \circ \tau_S, S)$ is the standard unknot by Lemma 4.5. It suffices to show that they are isotopic.

Certainly, the Legendrian $(W, \lambda, \tau_L^{-1} \circ \tau_S, S)$ is isotopic to $(W, \lambda, \tau_L^{-1} \circ \tau_S, (\tau_L^{-1} \circ \tau_S)(S))$ because the monodromy is a contactomorphism, and $(\tau_L^{-1} \circ \tau_S)(S) = \tau_L^{-1}(S) = L + S = \tau_S(L)$. $\square$

Theorems 4.1 and 4.2 prove $3a = 4$ and conclude the proof of Theorem 1.1.

5. Consequences

In this section we provide details for the proofs of the results stated in Subsections 1.3 and 1.4 in the introduction. The results are proven using the existence h–principle [**15**], Theorem 1.1 and the following theorem:

Theorem 5.1. *Consider $(\mathbb{R}^{2n+1}, \ker \alpha_{ot})$ a contact structure overtwisted at infinity. There exists a contact embedding $(\mathbb{R}^{2n+1} \times D^2(\varepsilon), \ker(\alpha_{ot} + \lambda_{st})) \longrightarrow (\mathbb{S}^{2n+3}, \xi_0)$, for $\varepsilon \in \mathbb{R}^+$ small enough. In particular, the contact manifold $(\mathbb{R}^{2n+1} \times D^2(\varepsilon), \ker(\alpha_{ot} + \lambda_{st}))$ is tight for $\varepsilon \in \mathbb{R}^+$ small enough.*

5.1. Proof of Theorem 5.1. In order to prove Theorem 5.1 we first show that there exists a unique contact structure overtwisted at infinity, up to proper isotopy, on $\mathbb{R}^{2n+1}$. This is proven in [**50**] for open smooth 3–folds and the argument readily adapts to higher dimensions.

Theorem 5.2. *Let Y be an open smooth manifold and ξ_1, ξ_2 two contact structures overtwisted at infinity. If ξ_1 and ξ_2 are homotopic as tangent complex hyperplane fields then they are properly isotopic.*

Proof. Consider an exhaustion $U_1 \subseteq U_2 \subseteq \ldots \subseteq Y$ by open smooth domains such that the inclusions are relatively compact and ξ_1 and

ξ_2 are overtwisted on $U_{i+1} \setminus \overline{U_i}$. Then we can use the uniqueness h–principle in [**15**] to apply a swindle argument with ξ_1 on $Op(\partial U_{2i})$ and ξ_2 on $Op(\partial U_{2i+1})$. On the one hand, isotoping relative to the boundaries $Op(\partial U_{2i})$ we can obtain the contact structure ξ_1. On the other, an isotopy relative to $Op(\partial U_{2i+1})$ yields ξ_2. $\square$

Let us now prove Theorem 5.1. Theorem 1.1 implies that

$$(\mathbb{R}^3 \times D(R_1, \dots, R_{n-1}), \ker(\alpha_{ot} + \lambda_{\mathrm{st}}))$$

is overtwisted if the radii $R_1, \dots, R_{n-1} \in \mathbb{R}^+$ are large enough. Since it is overtwisted at infinity, Theorem 5.2 provides a contactomorphism

$$f : (\mathbb{R}^{2n+1}, \ker \alpha_{ot}) \longrightarrow (\mathbb{R}^3 \times D(R_1, \dots, R_{n-1}), \ker(\alpha_{ot} + \lambda_{\mathrm{st}})).$$

Note first that there exists a proper contact embedding

$$\phi : (\mathbb{R}^3 \times D^2(\varepsilon), \ker(\alpha_{ot} + \lambda_{\mathrm{st}})) \longrightarrow (\mathbb{R}^5, \xi_0).$$

And second, the contact embedding ϕ extends to a contact embedding

$$\phi : (\mathbb{R}^3 \times D(\varepsilon, R_1, \dots, R_{n-1}), \ker(\alpha_{ot} + \lambda_{\mathrm{st}})) \longrightarrow (\mathbb{R}^{2n+3}, \xi_0).$$

The contact embedding $\phi \circ (f, id)$ proves the statement. $\square$

5.2. Proof of Theorem 1.4. Choose a number $s \in \mathbb{N}$ such that there exists a smooth proper embedding

$$i : Y \times D(R_1, \dots, R_{s-1}) \longrightarrow \mathbb{R}^{2n+2s-1}.$$

The existence h–principle [**15**] provides a contact structure $(\mathbb{R}^{2n+2s-1}, \ker \alpha)$ overtwisted at infinity such that $\xi|_{Y \times D(R_1,\dots,R_{s-1})} = \ker(\alpha_{ot} + \lambda_{\mathrm{st}})$. Then the contact embedding

$$(Y \times D(R_1, \dots, R_{s-1}, \varepsilon), \ker(\alpha_{ot}+\lambda_{\mathrm{st}})) \xrightarrow{(i,id)} (\mathbb{R}^{2n+2s-1} \times D^2(\varepsilon), \ker(\alpha+vdt))$$

and Theorem 5.1 imply that the contact manifold

$$(Y \times D(R_1, \dots, R_{s-1}, \varepsilon), \ker(\alpha_{ot} + \lambda_{\mathrm{st}}))$$

is tight $\forall R_1, \dots, R_{s-1} \in \mathbb{R}^+$ and $\varepsilon \in \mathbb{R}^+$ small enough. Choosing the radii $R_1, \dots, R_{s-1}$ large enough and applying Theorem 1.1, this implies that $Y \times D^2(\varepsilon)$ is tight for $\varepsilon \in \mathbb{R}^+$ small enough. $\square$

Remark 5.3. The argument given for Theorem 1.4 actually proves that the contact manifold $(Y \times D(R_1, \dots, R_{s-1}, \varepsilon), \ker(\alpha_{ot} + \lambda_{\mathrm{st}}))$ contact embeds in codimension–0 into a symplectically fillable contact manifold if $\varepsilon \in \mathbb{R}^+$ is small enough.

5.3. Proof of Theorem 1.1. Suppose that $\|H\|_{\mathcal{C}^0} < \delta^2$, then the results Chapter 6 in imply that $(Y \times D^2(\delta), \ker(\alpha + \lambda_{\mathrm{st}}))$ is PS–overtwisted. Hence $(Y \times D(R_1, \dots, R_{s-1}, \varepsilon), \ker(\alpha_{ot} + \lambda_{\mathrm{st}}))$ is GPS–overtwisted and this contradicts Theorem 1.4 and Remark 5.3 if $\delta \in \mathbb{R}^+$ can be chosen arbitrarily small. $\square$

5.4. Proof of Theorem 1.10. Suppose that $\sup\{c_f(p)\} < 1$ for all $p \in Y$ and consider the contact embedding

$$\psi_s : (Y \times D(R_1, \dots, R_s), \ker(\alpha_{ot} + \lambda_{\mathrm{st}})) \longrightarrow (Y \times D(R_1, \dots, R_s), \ker(\alpha_{ot} + \lambda_{\mathrm{st}}))$$
$$(p, r_1, \theta_1, \dots, r_s, \theta_s) \longmapsto (f(p), c_f^{1/2} r_1, \theta_1, c_f^{1/2} r_s, \theta_s).$$

Given any $\varepsilon \in \mathbb{R}^+$, choose $N \in \mathbb{N}$ such that the contact embedding ψ_s^N has image

$$\psi_s^N : (Y \times D(R_1, \dots, R_s), \ker(\alpha_{ot} + \lambda_{\mathrm{st}})) \longrightarrow (Y \times D(\varepsilon, \dots, \varepsilon), \ker(\alpha_{ot} + \lambda_{\mathrm{st}})).$$

Select the radii $R_1, \dots, R_s \in \mathbb{R}^+$ large enough such that the contact manifold

$$(Y \times D(R_1, \dots, R_s), \ker(\alpha_{ot} + \lambda_{\mathrm{st}}))$$

is overtwisted. Then the contact embedding

$$(Y \times D(R_1, \dots, R_s), \ker(\alpha_{ot} + \lambda_{\mathrm{st}})) \xrightarrow{\psi_{s+1}^N} (Y \times D(R_1, \dots, R_{s-1}, \varepsilon), \ker(\alpha_{ot} + \lambda_{\mathrm{st}}))$$

implies that the target manifold is overtwisted, thus contradicting Theorem 1.4. Hence there cannot exist such ψ_s and $\sup\{c_f(p)\} \geq 1$. $\square$

CHAPTER 4

Contact blow–up

In this fourth chapter we introduce the definitions of a contact blow–up from three different perspectives. The results were initially motivated by the use of Lefschetz type pencils in Chapter 2. The different approaches presented for the contact blow–up are related and we prove that the blown–up contact structures we obtain coincide in the case of blow–ups along transverse embedded loops. This is joint work with D.M. Pancholi and F. Presas.

1. Introduction

A contact structure ξ on a $(2n+1)$–dimensional manifold M is a codimension 1 tangent distribution which is maximally non–integrable. The distribution ξ can be locally defined as the kernel of a 1–form α, maximal non–integrability is tantamount to the condition $\alpha \wedge d\alpha^n \neq 0$. These structures naturally appear on the boundary of a large class of symplectic manifolds or on hypersurfaces therein. The study of contact structures on a manifold has significantly contributed to research in geometric topology. The techniques in 3–dimensional contact topology provide several knot invariants and also lead to 3–dimensional counterparts to the classical 4–dimensional gauge theories. In higher dimensions they are an essential ingredient in the study of Weinstein cobordisms and hence in the classification of symplectic structures on manifolds.

The existence of a contact structure on a manifold is a central question. This problem was first posed by S.S. Chern in 1966: find topological conditions for a smooth manifold in order that it admits a contact structure. A necessary topological condition is the reduction of the structure group of the tangent bundle to $U(n) \times \{1\}$, see [**61**]. Equivalently, the bundle ξ should admit a complex structure. Such a reduction is referred to as an almost contact structure. In general, the sufficiency of this condition is an open question. The first tour–de–force in this direction was due to M. Gromov. The h–principle techniques developed in [**73**] imply that any open manifold M with an almost contact structure admits a contact structure.

The situation for closed manifolds is quite different. The existence question has only been answered in special cases. In particular, any 3– or 5–dimensional orientable manifold admits a contact structure in a given homotopy class of complex hyperplanes. See [**94**], [**91**], [**57**] and Chapter 2 in this dissertation. Recent progress has been achieved in [**16**] for simply–connected 7–dimensional manifolds. In higher dimensions the question remains open.

The arguments to prove the above results have a common feature: smooth surgery techniques are adapted to the contact category. The contact structure is not directly constructed on the corresponding class of manifolds, instead we restrict the study to a subclass where suitable contact structures are known to exist. Then a series of topological surgeries is performed. The contact structure extends along them and thus contact structures are obtained in the larger class. Several surgery operations have been used, see [**63**]:

1. Handle–body attachments of low index as in [**123**].
2. Connected sums of manifolds in [**102**], and branched covers.
3. Fibered connected sums along codimension–2 contact submanifolds.

There is another construction proposed in the book *Partial Differential Relations* [**73**]. In contrast with the operations mentioned above, it has neither been studied nor used. The construction extends the classical blow–up operation to the contact category, see Exercise (c) on page 343 in [**73**]. This Chapter develops this notion and related constructions. This operation contributes to a better understanding of the existence of contact structures. Note that the construction, for the case of transverse knots, is used in Chapter 2 in an essential manner to prove the existence of contact structures on 5–dimensional manifolds. In the following discussion we describe the contact blow–up construction.

Let M be a smooth manifold and $S \overset{e}{\hookrightarrow} M$ an embedded submanifold. The normal bundle of (S, e) in M will be denoted by $\nu_M(S)$. Recall that it is defined by the short exact sequence of smooth vector bundles over S

$$0 \longrightarrow TS \xrightarrow{e_*} TM|_S \longrightarrow \nu_M(S) \longrightarrow 0.$$

Given a complex vector bundle $E \longrightarrow M$, we denote by $\mathbb{P}(E)$ the fiber-wise projectivization of E.

Suppose that the normal bundle $\nu_M(S)$ is a complex bundle. Then we may produce a manifold $\widetilde{M}$, the topological blow–up of M along S. It is defined as the connected sum

$$\widetilde{M} := M\#_S\overline{\mathbb{P}(\nu_M(S) \oplus \mathbb{C})}$$

of the manifolds M and $\mathbb{P}(\nu_M(S) \oplus \mathbb{C})$ with the reversed orientation along S. Let σ_0 be the zero section of $\nu_M(S)$. The submanifold S is embedded in the first factor through e and in the second as the section

$$\begin{array}{rcl} s : S & \longrightarrow & \mathbb{P}(\nu_M(S) \oplus \mathbb{C}) \\ p & \longmapsto & \langle(\sigma_0 \oplus 1)\rangle. \end{array}$$

In the category of symplectic manifolds the normal bundle is a complex bundle and the manifold $\widetilde{M}$ can be endowed with a symplectic structure. In this paper we address the corresponding question for contact manifolds.

In the above reference, M. Gromov conjectured that there exists a contact blow–up construction along a contact submanifold S embedded in a contact manifold M provided a pair of hypotheses are satisfied. These are:

H1. The contact submanifold $(S, \alpha_S = e^*(\alpha))$ is a Boothby–Wang manifold. See Definition 3.1. In particular, the Reeb vector field associated to α_S has all its orbits periodic with the same period. Let W be the quotient space of its orbits and $\pi : S \longrightarrow W$ the projection map.

H2. The normal symplectic vector bundle $\nu_M(S)$ is isomorphic to the pull–back of a symplectic vector bundle $V \longrightarrow W$ through π. That is, there exists an isomorphism $\nu_M(S) \cong \pi^*V$ of symplectic vector bundles.

These two hypotheses would allow to give a definition of the contact blow–up. Nevertheless the contact blow–up will not be a contact structure on the topological blow–up $\widetilde{M}$ of M. We will first illustrate a reason for this in a simple example, see Section 2. We provide a definition producing a contact structure on a manifold constructed as a different connected sum with M. It has the same geometrical properties as the symplectic blow–up. It is rather this manifold that we will call the contact blow–up.

This paper is organized as follows: Section 2 provides a brief review of the topological blow–up. In Section 3, we introduce the classical Boothby–Wang construction [**17**]. It will be described with some concrete examples that shall be used later on. Then, three alternative constructions of contact blow–up are introduced:

1. The contact blow–up for embedded transverse loops, produced as a surgery operation. This had been introduced in Chapter 2, but it is reviewed in Section 4 in order to provide context.
2. The contact blow–up defined *à la Gromov* is the content of Section 5.
3. The contact blow–up as a contact quotient is described in Section 6.

These three constructions are inspired by the three alternative constructions for the symplectic blow–up: the *ad hoc* construction with explicit gluings, the description using frame bundles, found on pages 239 and 243 in [**100**] respectively, and the symplectic cut procedure discussed in [**90**]. See also [**109**]. Finally, Section 7 relates these constructions in the case of transverse loops.

Acknowledgements. I would like to acknowledge K. Niederkrüger for useful discussions on Chapter 4, in particular for asking about the relation between the contact cut and the contact blow–up.

2. Preliminaries

In this section we introduce the basic definitions, explain the topological blow–up procedure and discuss an example.

DEFINITION 2.1. A contact structure on a smooth manifold M^{2n+1} is a maximally non–integrable smooth field ξ of tangent hyperplanes.

A contact manifold (M, ξ) is a choice of a contact structure ξ on M. The maximal non–integrability can be described in terms of local equations for ξ. A smooth field ξ of tangent hyperplanes is maximally non–integrable if and only if for any $p \in M$ there exist an open subset

$U \subset M$ containing p and a 1–form $\alpha \in \Omega^1(U)$ such that $\xi|_U = \ker \alpha$ and $\alpha \wedge d\alpha^n \neq 0$. Equivalently, the form $d\alpha$ is non–degenerate when restricted to ξ. In case the form α can be chosen to be globally defined, i.e. $\alpha \in \Omega^1(M)$, the contact structure ξ is called coorientable. A contact structure is cooriented if a choice of global contact form has been made.

Let (M, ξ) be a cooriented contact manifold with fixed global contact form α, i.e. $\alpha \in \Omega^1(M)$ satisfies $\ker \alpha = \xi$, $\alpha \wedge d\alpha^n \neq 0$. A smooth submanifold $S \overset{e}{\hookrightarrow} M$ is called a *contact submanifold* if the induced distribution $\xi_S = e^*(\xi)$ is a contact structure on S.

The notion of a blow–up has its origins in algebraic geometry. First, we define the concept for a complex vector space. See [**77**] for further details.

Definition 2.2. The blow–up $\widetilde{\mathbb{C}}_0^{n+1}$ of the complex vector space $\mathbb{C}^{n+1}$ at the origin is the smooth manifold $\mathcal{O}(-1) = \{([l], p) : p \in l\} \subset \mathbb{CP}^n \times \mathbb{C}^{n+1}$ along with the restriction of the projection onto the second factor $\sigma : \mathcal{O}(-1) \longrightarrow \mathbb{C}^{n+1}$.

Note that σ restricted to $\mathcal{O}(-1) \setminus \{([l], p) : p = 0\}$ induces a diffeomorphism onto the image $\mathbb{C}^{n+1} \setminus \{0\}$. The projective space $\sigma^{-1}(\{0\}) = \mathbb{CP}^n$ is called the exceptional divisor. The topological blow–up of M along S defined in the previous discussion coincides with the previous definition if $S = \{0\}$ is the origin in $M = \mathbb{C}^{n+1}$. More generally, from the definition of $\widetilde{M}$ we conclude the following

Lemma 2.3. *Let M be a smooth manifold and (S, e) a submanifold with complex normal bundle. There exists a smooth submanifold $E_S \subset \widetilde{M}$ diffeomorphic to the total space of a projective smooth bundle over S such that, as smooth manifolds, $M \setminus S \cong \widetilde{M} \setminus E_S$.*

The topological blow–up can be performed along any complex submanifold S of a complex manifold M. In this case the blown–up manifold $\widetilde{M}$ inherits a canonical complex structure. Analogously, if (M, ω) is a symplectic manifold and S a symplectic submanifold, the topological blow–up manifold $\widetilde{M}$ can also be endowed with a symplectic structure. In the symplectic case there is no uniqueness, see [**100**]. The topological blow–up can also be performed along a contact submanifold of a contact manifold because the normal bundle is symplectic and hence it is also complex.

REMARK 2.4. 1. Suppose the normal bundle $\nu_M(S)$ splits as a direct sum of isomorphic complex line bundles L: $\nu_M(S) = L \oplus \overset{\times r}{\cdots} \oplus L$. Then there is a second projection map $\pi_2 : \nu_M(S) \setminus S \longrightarrow \mathbb{CP}^{r-1}$ defined as follows. Given a point $p \in S$, let $s_p \in L_p$ be a non–zero vector in the fiber. Then a point $(l_1, \cdots, l_r) \in \nu_M(S)_p \setminus \{p\}$ is mapped to $\pi_2(l_1, \cdots, l_r) = [l_1/s_p : \cdots : l_r/s_p]$. It is simple to verify that the map is well–defined, i.e. independent of the choice of vector s_p.

2. The hypothesis above is satisfied in some cases. For instance, let S be the base locus of a projective, resp. symplectic, Lefschetz pencil. Then S conforms the hypothesis for $r = 2$. In this case the fibers of π_2 are projective, resp. symplectic. This also occurs with contact pencils, see [**112**].

Example: *Let (M^5, ξ) be a 5–dimensional contact manifold and let S be a 1–dimensional compact contact submanifold, i.e. a transverse embedded loop. If we perform a topological blow–up along S, the exceptional divisor is $E \cong \mathbb{S}^1 \times \mathbb{CP}^1 \cong \mathbb{S}^1 \times \mathbb{S}^2$. We are in the situation of the previous Remark: $\nu(S^1)$ is trivial. Therefore we have a projection $\pi_2 : \mathbb{S}^1 \times \mathbb{S}^2 \to \mathbb{S}^2$. In the contact case, if we assume that E is a contact submanifold, it is not possible to ensure that the fibers of such projection map are contact: there is no contact distribution on $\mathbb{S}^1 \times \mathbb{S}^2$ such that each $\mathbb{S}^1$ is transverse to the contact structure, see* [**70**].

In the previous example, the non–transversality of the fibers occurs only because we are using the topological blow–up as our blown–up manifold. We will further argue from different perspectives that the blown–up manifold $\widetilde{M}$ we should consider in contact topology is not the topological blow–up discussed above. Instead, the correct manifold is obtained through a procedure that substitutes $S \cong \mathbb{S}^1$ by the standard contact sphere $\mathbb{S}^3$, not by $\mathbb{S}^1 \times \mathbb{S}^2$. In such a case, the natural projection map $\pi : \mathbb{S}^3 \longrightarrow \mathbb{CP}^1$ is the Hopf fibration, whose fibers are transverse to the contact structure.

3. Boothby–Wang Constructions

In this section we explain the construction of a contact manifold from an integral symplectic manifold as developed in [**17**]. It will be used to understand the contact structure on the manifold obtained after a contact blow-up.

A symplectic manifold (W, ω) is called integral if the class $[\omega]$ lies in the image of the map $H^2(W, \mathbb{Z}) \longrightarrow H^2(W, \mathbb{R})$, i.e. the periods of ω are integers. Such a form ω is called integral. For instance, a Kähler form on a complex compact manifold is integral if and only if the manifold is a smooth projective algebraic variety. The constants have been normalized such that the unit disk has area 1. Note that the integral lift of $[\omega]$ to $H^2(W, \mathbb{R})$ may not be unique if $H^2(W, \mathbb{Z})$ contains torsion elements.

Given an integral form ω there exists a Hermitian complex line bundle L_ω over W admitting a compatible connection 1–form whose curvature is $-i\omega$. See [**18**] for the details. This leads to the following

Definition 3.1. Let (W, ω) be an integral symplectic manifold. The Boothby–Wang manifold $\mathbb{S}_k(W)$ is the contact manifold whose total space is the unit circle bundle associated to the line bundle $L_{k\omega}$ and its contact structure is defined as the restriction of any connection 1–form α with curvature form $d\alpha = -ik\omega$ to the circle bundle.

Remark 3.2. The contact structure is independent of the choice of the connection 1–form. Indeed, the space of choices for a connection 1–form as above is an affine space modelled on the vector space of flat connections and hence is contractible. Gray stability applies to ensure the uniqueness up to contactomorphisms of the contact structure.

For the case $k = 1$ we will sometimes omit the subindex k. Note that the topology of the total space varies with the parameter k. The exact relationship between the topology and the parameter k is the content of the following

Lemma 3.3. *Let (W, ω) be a symplectic manifold. Then the Boothby–Wang manifold $\mathbb{S}_1(W)$ is a k–fold covering of $\mathbb{S}_k(W)$.*

Proof. We fix a Hermitian connection on L. It induces a Hermitian connection on $L^{\otimes k}$. Define the unitary non–linear map between line bundles

$$L \longrightarrow L^{\otimes k}, \quad u \longmapsto u^{\otimes k}.$$

It preserves the connections on the two bundles. There exists a unitary connection–preserving action of $\mathbb{Z}_k$, the cyclic group of order k, on L given by

$$\mathbb{Z}_k \times L \longrightarrow L, \quad (c; u) \longmapsto e^{2\pi i c/k} u.$$

This action induces the trivial action on $L^{\otimes k}$ and it is the deck transformation group of a covering

$$\pi : \mathbb{S}(L) \longrightarrow \mathbb{S}(L^{\otimes k})$$

between the total spaces of the circle bundles associated to L and $L^{\otimes k}$. This map π is certainly compatible with the contact structures. $\square$

Examples: 1. Let $L(k; 1, \ldots, 1)$ be a lens space, i.e. the orbit space of the action

$$\mathbb{Z}_k \times \mathbb{S}^{2n-1} \longrightarrow \mathbb{S}^{2n-1}, \quad 1 \cdot (z_1, \ldots, z_n) = (e^{2\pi i/k} z_1, e^{2\pi i/k} z_2, \ldots, e^{2\pi i/k} z_n).$$

The lens space naturally inherits a contact structure ξ_L from the standard contact structure of $\mathbb{S}^{2n-1}$ induced by the complex tangencies. Lemma 3.3 provides a contactomorphism between $\mathbb{S}_k(\mathbb{CP}^{n-1})$ and the contact manifold $(L(k; 1, \ldots, 1), \xi_L)$.

2. Consider the 2–torus $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$ and τ an integral area form with total area one. Then the Boothy–Wang manifolds $\mathbb{S}_k(T^2)$ associated to (T^2, τ) give rise to quotients of the Heisenberg group by discrete subgroups Γ_k and thus provide examples of contact nilmanifolds different from the 3–torus.

The construction of the contact blow–up will involve the quotient of the product of two Boothby–Wang manifolds. We therefore proceed to describe the Boothby–Wang construction when the base symplectic manifold is a product. We show that the Boothby–Wang construction and the Cartesian product *commute*. Let $\mathbb{S}_{(b,a)}(W_1 \times W_2)$ be the Boothby–Wang manifold associated to

$$(W_1 \times W_2, b\pi_1^*\omega_1 + a\pi_2^*\omega_2),$$

then we have the following:

Theorem 3.1. *Let (W_1, ω_1) and (W_2, ω_2) be integral symplectic manifolds and $a, b \in \mathbb{Z}$ a pair of coprime integers. Consider the product $\mathbb{S}(W_1) \times \mathbb{S}(W_2)$ of the Boothby–Wang manifolds and the action*

$$\varphi_{(a,-b)} : \mathbb{S}^1 \times \mathbb{S}(W_1) \times \mathbb{S}(W_2) \longrightarrow \mathbb{S}(W_1) \times \mathbb{S}(W_2)$$

$$(p, q) \longmapsto \theta \cdot (p, q) = (a\theta \cdot p, -b\theta \cdot q)$$

Then the space of orbits is a manifold diffeomorphic to $\mathbb{S}_{(b,a)}(W_1 \times W_2)$*. This space of orbits carries a contact structure induced by a connection with curvature*

$$b\pi_1^*\omega_1 + a\pi_2^*\omega_2$$

and hence is contactomorphic to $\mathbb{S}_{(b,a)}(W_1 \times W_2)$*.*

PROOF. Let $G = \mathbb{S}^1 \times \mathbb{S}^1$ and $H \cong \mathbb{S}^1 \subset G$ be the subgroup defined as the image of the embedding

$$\varphi_{(a,-b)} : \mathbb{S}^1 \longrightarrow H \subset G, \quad \sigma \longmapsto (a\sigma, -b\sigma).$$

Let P be the G–principal bundle with base space $W_1 \times W_2$ induced by the $\mathbb{S}^1$–principal bundles $\mathbb{S}_1(W_1)$ and $\mathbb{S}_1(W_2)$. Our aim is to describe P/H as a bundle over $W_1 \times W_2$. In general $P \longrightarrow P/H$ is not a H–principal bundle but this is the case when both G and H are closed Lie groups and H is a normal sub-group of G. Actually, G and H are abelian and since $(a, b) = 1$, P/H is also a G/H–principal bundle over $W_1 \times W_2$. Taking into account the exact group sequence

$$1 \longrightarrow \mathbb{S}^1 \cong H \longrightarrow G \longrightarrow G/H \cong \mathbb{S}^1 \longrightarrow 1$$

where the second morphism is given by multiplication by (b, a), we conclude that the space of orbits P/H is a manifold diffeomorphic to $\mathbb{S}_{(b,a)}(W_1 \times W_2)$. The claim about the connection and the associated curvature follows from the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{(a,-b)} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(b,a)^t} \mathbb{Z} \longrightarrow 0.$$

Finally, it follows from Remark 3.2 that the two manifolds are, in fact, contactomorphic. $\square$

There are a few simple cases worth mentioning.

Examples: 1. Let $W_1 = \{pt\}$ and W_2 arbitrary. Then neither the topology of the resulting space nor the contact structure depend on b. Indeed, $\mathbb{S}^1 \times \mathbb{S}_1(W_2)/\sim$ is diffeomorphic to

$$\mathbb{S}_{(b,a)}(pt \times W_2) \cong \mathbb{S}_a(W_2).$$

Analogously, the parameter a is vacuous if $W_2 = \{pt\}$. In particular, the quotient of $\mathbb{S}^1 \times \mathbb{S}^1$ by any $(a, -b)$ coprime $\mathbb{S}^1$–action is diffeomorphic to $\mathbb{S}^1$.

2. Let $W_1 = W_2 = \mathbb{CP}^1$ be symplectic manifolds with the Fubini–Study form. Then the space $\mathbb{S}_{(b,a)}(\mathbb{CP}^1 \times \mathbb{CP}^1)$ is diffeomorphic to $\mathbb{S}^3 \times \mathbb{S}^2$ regardless of the values $a, b \in \mathbb{N}$, see [**120**] for a proof of this fact. Further, the symplectic structure of the associated line bundle depends only on $a - b$. Note that there is an alternative construction of a contact structure on $\mathbb{S}^3 \times \mathbb{S}^2$ using an open book decomposition with $T^*\mathbb{S}^2$ pages and an even power of a Dehn twist as monodromy. However, such a procedure may only produce vanishing first Chern class and is thus different from $\mathbb{S}_{b,a}(\mathbb{CP}^1 \times \mathbb{CP}^1)$ if $a \neq 1$. See [**86**] for more details.

3. The previous example can be generalized to construct contact structures on $\mathbb{S}^{2n+1} \times \mathbb{S}^2$. Indeed Theorem 3.1 implies that the total space of $\mathbb{S}_{(1,k)}(\mathbb{CP}^n \times \mathbb{CP}^1)$ is an $\mathbb{S}^{2n+1}$–bundle over $\mathbb{S}^2$. The Hopf action is explicit enough for the classifying map to be described as the element

$$(n+1)k \in \mathbb{Z}_2 \cong \pi_1(SO(2n+2)).$$

Consequently the resulting manifold is diffeomorphic to $\mathbb{S}^{2n+1} \times \mathbb{S}^2$ if n is odd or k is even.

It will be essential for the contact blow–up construction to be able to extend a connection on a submanifold to a global connection. Let us now prove that this is possible under suitable conditions:

LEMMA 3.4. *Let S be a closed submanifold of (W^{2n}, ω), possibly with smooth boundary, and L the line bundle associated to ω. Assume that the restriction morphism $H^1(W) \longrightarrow H^1(S)$ is surjective and let A_S be a connection over $L|_S$ whose curvature is $-i\omega$. Then there is a connection A on L with curvature $-i\omega$ such that its restriction to S is A_S.*

PROOF. Let A_0 be a connection on the line bundle $L \longrightarrow W$ with curvature $-i\omega$. Denote $i : S \longrightarrow W$, then $A_S - i^*A_0 = \beta_S$ is a closed 1–form over S. In order to complete our argument we need to extend β_S to a global closed 1–form.

By hypothesis the map $H^1(W) \longrightarrow H^1(S)$ is a surjection. Therefore there exists a cohomology class $[\beta]$ on $H^1(W)$ whose restriction to S coincides with $[\beta_S]$. Its difference over S will be the trivial class on $H^1(S)$, so $\beta_S - i^*\beta = dH_S$, for some smooth function $H_S : S \longrightarrow \mathbb{R}$. We extend H_S to a global smooth function $H : W \longrightarrow \mathbb{R}$. The form $A_0 + \beta + dH$ is the required global connection with curvature $-i\omega$ and extending A_S. $\square$

4. Surgery along transverse loops

Let (M^{2n+1}, ξ) be a contact manifold. In this section we recall the blow–up construction from Section 5 in Chapter 2. This is an operation defined in a neighborhood of a transversely embedded loop. Topologically it consists of a surgery along the loop: the interior of $\mathbb{S}^1 \times B^{2n}$ is removed and a tubular neighbourhood of the $(2n-1)$–sphere $B^2 \times \mathbb{S}^{2n-1}$ is glued along the common boundary $\mathbb{S}^1 \times \mathbb{S}^{2n-1}$. The sphere $\{0\} \times \mathbb{S}^{2n-1}$ whose neighbourhood is attached is called the exceptional divisor. Let us discuss this surgery operation in the contact category.

Consider the manifold $T = \mathbb{S}^1 \times (0,1) \times \mathbb{S}^{2n-1}$ with spherical coordinates (θ, r, σ). Let $\alpha_{std} = (dr \circ i)|_{\mathbb{S}^{2n-1}}$ be the standard contact form for the contact structure

$$\xi = T\mathbb{S}^{2n-1} \cap i(T\mathbb{S}^{2n-1})$$

on the sphere $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$. Define the following two contact forms in T:

$$\eta = d\theta - r^2\alpha_{std}, \quad \lambda = r^2 d\theta + \alpha_{std}. \tag{4.1}$$

Fix an integer $l \in \mathbb{Z}$ and consider the diffeomorphism

$$\begin{array}{rccc} \phi_l : & T & \longrightarrow & T \\ & (\theta, r, z) & \longrightarrow & (\theta, r, e^{2\pi i l\theta} z) \end{array} \tag{4.2}$$

It pulls–back the contact form η to $\overline{\lambda} = (-r^2) \cdot [(l - r^{-2})d\theta + \alpha_{std}]$.

Given a subset $C \subset M$, let $\mathcal{U}(C)$ denote a small closed tubular neighbourhood of C in M. These ingredients suffice to prove the following:

THEOREM 4.1. *Let (M^{2n+1}, ξ) be a contact manifold. Let $S \subset M$ be a smooth transverse loop in M. There exists a smooth manifold $\overline{M}$ satisfying the following conditions:*

- *There exists a contact structure $\overline{\xi}$ on $\overline{M}$.*
- *There exists a codimension–2 contact submanifold E in $\overline{M}$ with trivial normal bundle. The manifold $(E, \overline{\xi})$ is contactomorphic to the standard contact sphere $(\mathbb{S}^{2n-1}, \xi)$.*
- *The manifolds $(M \setminus \mathcal{U}(S), \xi)$ and $(\overline{M} \setminus E, \overline{\xi})$ are contactomorphic.*

The manifold $(\overline{M}, \overline{\xi})$ will be called the contact surgery blow–up of M along S. The contact submanifold $(E, \overline{\xi})$ is called the exceptional divisor.

PROOF. By Gray stability, we may assume that a tubular neighbourhood of the embedded loop is contactomorphic to $\mathbb{S}^1 \times B^{2n}(\varepsilon)$ with the contact form η as in (4.1), for some small radius $r \leq \varepsilon$. We enlarge this tubular ε–neighbourhood using the squeezing technique from [**53**] to obtain a radius 2 neighbourhood. More precisely, we need the following auxiliary lemma:

LEMMA 4.1. (*Proposition 1.24 in* [**53**]) *Let $k > 0$ be a positive integer and $R_0 > 0$ a radius. Then the following map is a contactomorphism*

$$\begin{aligned} \psi_k : \mathbb{S}^1 \times B^{2n}(R_0) &\longrightarrow \mathbb{S}^1 \times B^{2n}\left(\frac{R_0}{\sqrt{1+kR_0^2}}\right) \\ (\theta, r, w_1, \ldots, w_n) &\longrightarrow \left(\theta, \frac{r}{\sqrt{1+kr^2}}, e^{2\pi i k\theta} w_1, \ldots, e^{2\pi i k\theta} w_n\right), \end{aligned}$$

and it restricts to the identity at $\mathbb{S}^1 \times \{0\}$.

Consider $R_0 = 2$ in the lemma above, then we need k so large that

$$\frac{2}{\sqrt{1+4k}} < \varepsilon.$$

We may therefore assume that the tubular neighbourhood for which the standard equation (4.1) holds for η has radius $r = 2$. In the annulus corresponding to the radius interval $(3/2, 2)$ use ϕ_1 to induce the contact structure given by $\ker \overline{\lambda}$. Declare $\ker \lambda$ to define the contact structure in the radius interval $[0, 1/2]$. In view of (2) with $l = 1$ we are left to find a strictly increasing function interpolating between r^2 and $1 - r^{-2}$ in the middle region. This can be done, see Figure 1. $\square$

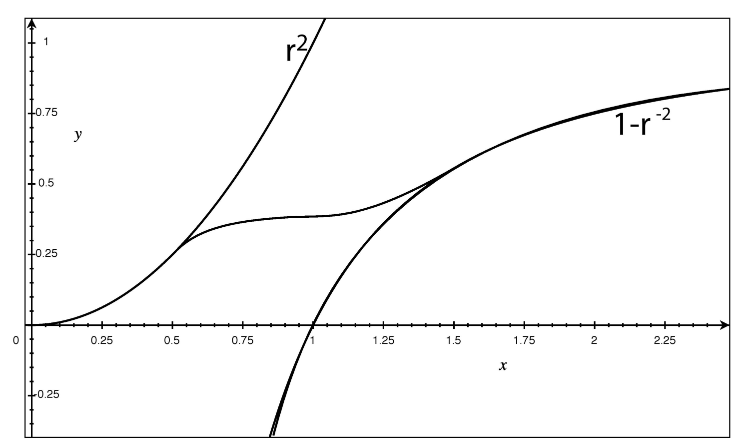


FIGURE 1. Interpolation matching λ and $\overline{\lambda}$.

Remark 4.2. The process described in the proof can be modified to include the radius squeezing in the gluing map. It suffices to use ϕ_l as gluing map instead of ϕ_1 in the domain. Indeed, denote $T_\rho = \mathbb{S}^1 \times (0,\rho) \times \mathbb{S}^{2n-1}$ and consider the contact structures

$$\xi_0 = \ker\{d\theta - r^2\alpha_{std}\}, \quad \xi_l = \ker\{(l - r^{-2})d\theta + \alpha_{std}\}.$$

Define the map

$$\varphi : T_2 \longmapsto T_{\varepsilon(k)}, \quad (\theta, r, z) \longmapsto \left(\theta, \frac{r}{\sqrt{1+kr^2}}, z\right),$$

where $\varepsilon(k)$ is the obvious radius in the image. Then the following diagram is commutative in the contact category :

$$\begin{tikzcd}
(T_2,\xi_1) \arrow[r, "\phi_1"] \arrow[d, "\varphi"] & (T_2,\xi_0) \arrow[d, "\psi_k"] \\
(T_{\varepsilon(k)},\xi_l) \arrow[r, "\phi_l"] & (T_{\varepsilon(k)},\xi_0)
\end{tikzcd}$$

where Lemma 4.1 is performed with parameter $k = l - 1$.

Note that the contactomorphism type of the exceptional divisor is that of the standard sphere. The parameter in the construction allows us to discretely vary the radius of the tubular neighbourhood we are collapsing. Suppose that $n \geq 2$.

Lemma 4.3. *The maps ϕ_l and ϕ_k are smoothly isotopic if and only if $(k-l)n$ is even.*

Proof. Let $t \in \mathbb{S}^1$ be the circle coordinate. Consider the morphism

$$\Psi : \pi_1(SO(2n)) \longrightarrow \pi_0(Diff(\mathbb{S}^1 \times \mathbb{S}^{2n-1})), \quad \Psi(\gamma_t)(\theta, z) = (\theta, \gamma_\theta(z)).$$

If γ_k denotes k–times the standard circle action on $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$ induced by $\mathbb{C}^*$, then it is clear that ϕ_k is realized as $\Psi(\gamma_k)$. Since $\pi_1(SO(2n)) \simeq \mathbb{Z}_2$ for $n \geq 2$ and $\gamma_k \simeq k \cdot n$ under this identification, $\gamma_k = \gamma_l$ if and only if $(k-l)n$ is even.

It remains to prove that ϕ_0 and ϕ_1 are not isotopic, for n odd. Construct two manifolds X_0 and X_1 by gluing two copies of the manifold $B^2 \times \mathbb{S}^{2n-1}$ respectively using ϕ_0 and ϕ_1 along the boundary. These manifolds are not diffeomorphic: a sphere is a spin manifold and the product formula for characteristic classes implies that so is $X_0 = \mathbb{S}^2 \times \mathbb{S}^{2n-1}$. On the other hand, the manifold X_1 is not spin. This can be seen by using any section s of the twisted bundle $X_1 \longrightarrow \mathbb{S}^2$, such s exists because

$n \geq 2$. Denote by $\nu(s(\mathbb{S}^2))$ the normal bundle to the section and let $E_1 \longrightarrow \mathbb{S}^2$ be the complex bundle over $\mathbb{S}^2$ such that $\mathbb{S}(E_1) = X_1$. Then $s^*(\nu(s(\mathbb{S}^2)) \oplus \mathbb{R}) = E_1$. Note that $w_2(E_1) = 1$ if n is odd and

$$w_2([s(\mathbb{S}^2)]) = w_2(TX_{1|s(\mathbb{S}^2)}) = w_2(\nu(s(\mathbb{S}^2)) = w_2(s^*(\nu(s(\mathbb{S}^2)) \oplus \mathbb{R})) = w_2(E_1).$$

Hence ϕ_0 and ϕ_1 are not isotopic. $\square$

In particular, for n odd the smooth type of the contact blow–up manifold will depend on the parity of the positive integer fixed for the construction. As for the contact type, it follows from Theorem 1.2 in [**53**] that the maps ϕ_k and ϕ_l are not contact compactly supported isotopic if $k \neq l$. This does not imply that the contact structures are different, but at least there is no local contactomorphism relating the two contact structures.

5. Gromov's approach

In this section we develop the contact blow–up along a Boothby–Wang submanifold, as suggested in [**73**]. The existence of a minimal radius for the tubular neighbourhood of the submanifold along which we will perform the blow–up will play an important role. This feature will be revisited in the definition provided in Section 6.

Let us review the definition of the symplectic blow–up, see [**100**] for more details.

5.1. Symplectic blow–up. Let (W, ω) be a symplectic manifold and S a symplectic submanifold of codimension $2k \geq 4$. Consider the symplectic normal bundle (ν_S, π) of S in M and fix a compatible almost complex structure ν_S. The choice of a compatible almost complex structure for a symplectic form induces a metric, and the equality

$$U(k) = O(2k) \cap \ Sp(2k, \mathbb{R})$$

implies that the structure group of ν_S can be considered to be $U(k)$. Thus ν_S is an associated vector bundle of a $U(k)$–principal bundle $P \longrightarrow S$.

The symplectic blow–up of W along S is obtained by the fiberwise symplectic blow–up of ν_S. Hence we require the analogue of Definition 2.2 for the case of symplectic vector spaces. Let $(\mathbb{R}^{2k}, \omega_0)$ be the standard symplectic vector space and ω_{FS} the standard Fubini–Study form on complex projective space. We will use the following

Definition 5.1. A symplectic blow–up of $(\mathbb{R}^{2k}, \omega_0)$ at the origin with radius δ is a symplectic manifold $(\widetilde{\mathbb{R}}^{2k}_\delta, \widetilde{\omega}_\delta)$ such that:

1. $\widetilde{\mathbb{R}}^{2k}_\delta \xrightarrow{\pi} \mathbb{R}^{2k}$ is a topological blow–up of $\mathbb{C}^k$ at the origin. The symplectic form induced on the exceptional divisor $E \cong \mathbb{CP}^{k-1}$ is $\delta^{\frac{1}{2k}} \cdot \omega_{FS}$.
2. For any $\varepsilon > 0$, there exists a symplectomorphism
$$\widetilde{\mathbb{R}}^{2k}_\delta \setminus \pi^{-1}(B(\delta+\varepsilon)) \cong \mathbb{R}^{2k} \setminus B(\delta+\varepsilon)$$
3. The unitary group $U(k)$ acts Hamiltonianly on $(\widetilde{\mathbb{R}}^{2k}_\delta, \widetilde{\omega}_\delta)$.

The symplectic blow–up of $(\mathbb{R}^{2k}, \omega_0)$ at the origin exists for each δ.

Remark 5.2. Note that the definition depends on δ. This parameter does not appear in Definition 2.2 since any linear homothety at the origin is a complex isomorphism.

Let us describe the non–linear symplectic blow–up of W along S. Property 3 in the above definition allows us to associate to P a bundle $(\widetilde{\nu}_{S,\delta}, \widetilde{\pi})$ over S with fiber $\widetilde{\mathbb{R}}^{2k}_\delta$. Let β be a connection in P and $\varepsilon > 0$, there are induced coupling forms α and $\widetilde{\alpha}_\delta$, in ν_S and $\tilde{\nu}_{S,\delta}$ respectively, restricting to the symplectic form on each fiber and coinciding away from the radius $\delta+\varepsilon$, see Thm. 6.17 in [**100**]. Define the forms

$$\begin{aligned} \omega_\nu &= \alpha + \pi^*\omega_S \\ \widetilde{\omega}_\nu &= \widetilde{\alpha}_\delta + \widetilde{\pi}^*\omega_S \end{aligned}$$

on the bundles ν_S and $\widetilde{\nu}_{S,\delta}$. These are symplectic forms close to the zero section and to the exceptional divisor respectively.

These forms also coincide away from a neighbourhood of S of radius $\delta+\varepsilon$. Let $U_{\delta_0} = P \times_{U(k)} B(\delta_0)$ be a neighbourhood of the zero section of the symplectic normal bundle. By the symplectic neighbourhood theorem there is a neighbourhood $\mathcal{U}(S)$ of the symplectic submanifold S and a symplectomorphism $\Psi : \mathcal{U}(S) \cong U_{\delta_0}$. Thus any fiberwise symplectic blow–up on ν_S with radius $0 \leq \delta+\varepsilon < \delta_0$ can be glued back to the initial manifold W using the symplectomorphism Ψ. The resulting manifold is the symplectic blow–up of W along S with radius δ.

Observe that the radius of the tubular neighbourhood of S cannot be estimated a priori. Therefore the symplectic volume of the exceptional divisor cannot be assumed to be arbitrarily large. This will be an obstruction to develop Gromov's approach in the contact category.

Example: Let V be a rank–$2k$ symplectic vector bundle over a symplectic manifold (W, ω). Then the total space is symplectic as well. Thus, we can blow–up the symplectic manifold V along its zero section W. In the case the symplectic form ω is integral, the symplectic form in the resulting blown–up manifold will be integral if the blow–up radius is $m^{\frac{1}{2k}}$, $m \in \mathbb{N}$. We call this a radius m blow–up.

5.2. Definition of Contact Blow–up. We now define the contact blow–up in terms of the symplectic blow–up. This is the second notion listed in Section 1.

Let (M, ξ) be a contact manifold and (S, ξ_S) a contact submanifold. We assume:

- H1. The contact submanifold S is contactomorphic to a Boothy–Wang manifold $\mathbb{S}(W, \omega)$.
- H2. Let $\pi : \mathbb{S}(W) \longrightarrow W$ be the circle bundle projection. There exists a symplectic vector bundle V over W such that, as symplectic vector bundles $\nu_M(S) \cong \pi^* V$.

The total space of V carries a symplectic form $\overline{\omega}$ in the same cohomology class $[\omega]$, under the natural identification of $H^2(V, \mathbb{R})$ with $H^2(W, \mathbb{R})$. As previously explained, there exists a symplectic manifold $(\widetilde{V}, \overline{\omega}_W)$ obtained by blowing up V along its zero section W. Suppose that the parameter multiplying the class of the exceptional divisor E in the symplectic blow–up is a positive integer, i.e. the symplectic form on $\widetilde{V}$ is integral.

The construction of the contact blow–up is based on the following diagram:

$$\begin{tikzcd}
\nu_M(S) \cong \pi^*(V) \arrow[d] & \mathbb{S}(V) \arrow[d] & \mathbb{S}(\widetilde{V}) \supset \mathbb{S}(E) = \mathbb{S}(\widetilde{V})|_E \arrow[d] \\
(S, \xi_S) \cong \mathbb{S}(W) \arrow[d, "\pi"'] & (V, \overline{\omega}) \arrow[dl] & (\widetilde{V}, \overline{\omega}_W) \supset E \arrow[l] \\
(W, \omega) & &
\end{tikzcd}$$

Diagram 1. Contact Blow–up Setup

Except for the blow–up projection $\widetilde{V} \longrightarrow V$, each map is a bundle projection. It is essential to understand the relation between the contact manifolds $\mathbb{S}(W)$, $\mathbb{S}(V)$ and $\mathbb{S}(E)$. This is the content of the following:

Lemma 5.3. *In the hypotheses above, $\mathbb{S}(W)$ is a contact submanifold of $\mathbb{S}(V)$. There are contactomorphic neighbourhoods $\mathcal{U}(\mathbb{S}(W))$ and $\mathcal{U}(S)$ in $\mathbb{S}(V)$ and M respectively.*

Proof. The choice of symplectic form on V implies that there exists a symplectic embedding of W in V and therefore $\mathbb{S}(W)$ is contained in $\mathbb{S}(V)$ as a contact submanifold. The tubular neighbourhood theorem states that the normal bundle $\nu_M(S)$ is diffeomorphic to a small neighbourhood of S in M, but $\nu_M(S) \cong \pi^*(V)$ so the same situation applies to $\mathbb{S}(W)$ in $\mathbb{S}(V)$. The last statement now follows from the contact neighbourhood theorem. □

As a consequence, $\mathbb{S}(W) \subset \mathbb{S}(V)$ provides a local model for $S \subset M$. Thus we only need to perform the blow–up of V along W and study whether the Boothby–Wang structures associated to them allow us to glue back the resulting blown–up model to M. This is the content of the following:

Proposition 5.4. *Let $S = \mathbb{S}(W)$ be a Boothby–Wang contact submanifold of $\mathbb{S}(V)$. Suppose we symplectically blow–up $W \subset V$ by collapsing a radius 1 neighbourhood. Then, there is a choice of contact form on $\mathbb{S}(\widetilde{V})$ such that $\mathbb{S}(E)$ is a contact submanifold of $\mathbb{S}(\widetilde{V})$ and the complement of a sufficiently small neighbourhood of $\mathbb{S}(E)$ in $\mathbb{S}(\widetilde{V})$ is contactomorphic to the complement of some neighbourhood of $\mathbb{S}(W)$ in $\mathbb{S}(V)$.*

For the sake of a clearer exposition the proof is explained at the end of this subsection.

Suppose we can choose a tubular neighbourhood $\mathcal{U}(\mathbb{S}(W)) \subset \mathbb{S}(V)$ with radius larger than 1 which is contactomorphic to a tubular neighbourhood $\mathcal{U}(S) \subset M$. Then we can make the following

DEFINITION 5.5. The contact blow–up of (M, ξ) along (S, ξ_S) is the contact manifold (M', ξ') obtained by removing the neighbourhood $\mathcal{U}(S)$ and gluing along its boundary a small neighbourhood of $\mathbb{S}(E)$ in $\mathbb{S}(\widetilde{V})$.

The contact manifold (M', ξ') is contactomorphic to M away from small neighbourhoods of $\mathbb{S}(E)$ and S respectively. The *exceptional divisor* of the contact blow–up is defined to be $\mathbb{S}(E)$, where E is the exceptional divisor of the symplectic blow–up over which it is locally modelled. Observe that for the definition to work we need S to have a tubular neighbourhood of radius at least 1 inside M.

Example: 1. The simplest example of contact blow–up is the case of a transverse loop K in (M^5, ξ). The loop is contactomorphic to $\mathbb{S}(pt)$ and its normal bundle is the pull–back of the trivial bundle over the point. Thus H1 and H2 are satisfied. The symplectic model corresponds to the blow–up of $\mathbb{C}^2$ at the origin, collapsing a neighbourhood of radius 1, and therefore $E = \mathbb{CP}^1$. Hence, $\mathbb{S}(E) = \mathbb{S}(\mathbb{CP}^1)$, i.e. the standard contact 3–sphere. This particular case can be seen, at least topologically, as a surgery along a loop.

2. In the previous example we may symplectically blow–up with radius $k \in \mathbb{N}$. The exceptional divisor is then $\mathbb{S}(\mathbb{CP}^1, k\omega_{\mathbb{CP}^1})$, i.e. the sphere bundle associated to the polarization $\mathrm{SO}(k)$ of $\mathbb{CP}^1$, which is the lens space $L(k; 1)$ with its standard contact structure. Therefore, even the diffeomorphism type of the blown–up contact manifold changes with the blow–up radius $k \in \mathbb{N}$.

Note that there is no natural projection map from $\mathbb{S}(E)$, the exceptional divisor, to the blow–up locus $\mathbb{S}(W)$. In the case of a loop in a 5–dimensional manifold, the exceptional divisor for a radius 1 blow–up is $\mathbb{S}^3$ and the blow–up locus is the circle $\mathbb{S}^1$. This is a difference with respect to the symplectic and algebraic cases where the exceptional divisor is a bundle over the submanifold along which the blow–up is performed.

It is true though that there is a natural projection $\mathbb{S}(E) \longrightarrow E \longrightarrow W$, but it does not lift to $\mathbb{S}(W)$.

Remark 5.6. The assumption of integral radius can be fulfilled in certain cases. For instance in the blow–up along a transverse $\mathbb{S}^1$ we can use Lemma 4.1. Therefore the construction in this case will have two natural parameters: the integral radius that determines the topology of the exceptional divisor, and the choice of framing in the spirit of Lemma 4.1. In the construction à la Gromov described above, the second positive integer does not appear, in general. This is one reason why we introduce in the next section a third definition of the contact blow–up, highlighting these two choices.

To conclude this subsection we prove the assertion that allowed us to glue the Boothby–Wang construction over the exceptional divisor in the contact blow–up construction.

Proof of Proposition 5.4. We need to find an appropriate connection on the topological Boothby–Wang manifold over $\widetilde{V}$.

By the construction of the symplectic blow–up as given in [**100**], we know that given a sufficiently small neighbourhood of E in $\widetilde{V}$ one can choose a symplectic form ϖ on $\widetilde{V}$ such that the complement of that neighbourhood in $\widetilde{V}$ is symplectomorphic to a small neighbourhood of W in V. The exceptional divisor E is just the inverse image of W contained in V as the zero section under the blow–up projection $\phi : \widetilde{V} \longrightarrow V$.

Recall from Definition 3.1 that the contact structure of $\mathbb{S}(\widetilde{V})$ is determined by the choice of a connection form on the associated line bundle whose curvature is $-i\varpi$. So let A be the connection form on L that determines the contact structure on $\mathbb{S}(V)$, and denote by U an arbitrarily small neighbourhood of W inside V. From the construction of the symplectic form ϖ on $\widetilde{V}$ we can assume that the map ϕ is a symplectomorphism between $V \setminus U$ and $\phi^{-1}(V \setminus U)$. Therefore the connection $\phi^*(A)$ satisfies the required properties on $\phi^{-1}(V \setminus U)$. It remains to extend $\phi^*(A)$ to a connection over all of $\widetilde{V}$ with curvature $-i\varpi$. By Lemma 3.4 this is possible provided that the restriction morphism $H^1(\phi^{-1}(V \setminus U), \mathbb{R}) \longrightarrow H^1(\widetilde{V}, \mathbb{R})$ is surjective. For this it suffices to show that the inclusion induces an isomorphism $\pi_1(\widetilde{V}) \cong \pi_1(\phi^{-1}(V \setminus U))$.

Indeed, observe that $\widetilde{V}$ is homotopy equivalent to a $(\mathbb{CP}^{r-1})$–bundle over W, with $r \geq 2$, and hence $\pi_1(\widetilde{V}) = \pi_1(W)$ holds. Note that the manifold $\phi^{-1}(V \setminus U)$ is diffeomorphic to $V \setminus U$ and the set $V \setminus U$ is homotopy equivalent to a sphere bundle over W with fibers of dimension ≥ 3. From the long exact sequence of homotopy groups we conclude that

$$\pi_1(\phi^{-1}(V \setminus U)) \cong \pi_1(W).$$

Therefore Lemma 3.4 applies and there is a choice of contact form on $\mathbb{S}(\widetilde{V})$ with the required properties. □

6. Blow–up as a quotient

In this section we define the contact blow–up of a contact manifold M along a Boothby–Wang contact submanifold S using the notion of contact cuts.

6.1. Contact cuts. Given an $\mathbb{S}^1$–action on a manifold M, topologically the cut construction is based on collapsing the boundary of a tubular neighborhood of a given submanifold invariant by the action. Basic knowledge on the contact reduction procedure is assumed in the next few paragraphs, see [**61**]. Let us recall the construction of the contact cut for a contact $\mathbb{S}^1$–action as developed by E. Lerman:

THEOREM 6.1. (*Thm. 2.11 in* [**89**]) *Let* $(M, \ker\alpha)$ *be a contact manifold with an* $\mathbb{S}^1$*–action preserving* α *and let* μ *denote its moment map. Suppose that* $\mathbb{S}^1$ *acts freely on the zero level set* $\mu^{-1}(0)$*. Then the set*[1]

$$M_{[0,\infty)} := \{m \in M | \mu(m) \in [0, \infty)\}/\sim$$

is naturally a contact manifold. Moreover, the natural embedding of the reduced space

$$M_0 := \mu^{-1}(0)/\mathbb{S}^1$$

into $M_{[0,\infty)}$ *is contact and the complement* $M_{[0,\infty)} \setminus M_0$ *is contactomorphic to the open subset*

$$\{m \in M | \mu(m) > 0\} \subset (M, \ker\alpha).$$

REMARK 6.1. Note that contact reduction requires the regular value to be 0, whereas in symplectic reduction any regular value is licit. This is so because in contact reduction it is imposed that the orbits of the isotropy subgroup are tangent to the contact structure, see [**63**].

[1] The equivalence relation is defined as $m \sim m' \Longrightarrow \mu(m) = \mu(m') = 0$ and $m = \theta \cdot m'$ for some $\theta \in \mathbb{S}^1$.

6.2. Blow–up procedure. Let (M^{2n+1}, ξ) be a contact manifold and $(S, \ker\alpha)$ a codimension–$2k$ contact submanifold. Suppose that $(S, \ker\alpha) \cong \mathbb{S}_a(W)$ for some symplectic manifold (W, ω), $a \in \mathbb{N}$, and that ν_S is the trivial rank–k complex vector bundle over S. We will define the *contact blow–up of M along S*.

Remark 6.2. Any isocontact embedding[2] of a contact 3–fold into a sphere has trivial normal bundle. This situation does occur: any closed cooriented 3–fold admits an isocontact embedding into the standard contact 7–sphere. See [**73**] for an h–principle providing such isocontact embeddings.

A tubular neighbourhood of the contact submanifold S is contactomorphic to

$$S_R = S \times B^{2k}(R) \overset{sph.coord.}{\twoheadleftarrow} S \times [0, R) \times \mathbb{S}^{2k-1}, \quad \text{for some } R > 0,$$

with the contact structure given by $\alpha + r^2\alpha_{std}$, where α_{std} is the standard contact form on $\mathbb{S}^{2k-1}$. Let $b \in \mathbb{N}$ and consider the $\mathbb{S}^1$–action

$$\varphi_{(a,-b)} : \mathbb{S}^1 \times S \times [0, R) \times \mathbb{S}^{2k-1} \longrightarrow \mathbb{S}(W) \times [0, R) \times \mathbb{S}^{2k-1}$$

$$(\theta, p, r, z) \longmapsto ((a\theta) \cdot p, r, e^{-2\pi i b\theta} z).$$

This action is generated by the field $X = aR_S - bR_{std}$ where R_S, R_{std} are the Reeb vector fields associated to α and α_{std}. The moment map of the above action is

$$\mu_{(a,b)} : S \times B^{2k}(R) \longrightarrow \mathfrak{g}^* \cong \mathbb{R}$$

$$(p, r, z) \longmapsto a - br^2.$$

The contact cut can only be performed in the pre–image of the regular value $0 \in \mathbb{R}$, it is thus a necessary condition that $R^2 \geq a/b$. This can always be achieved if b is large enough.

Definition 6.3. Let $S \cong \mathbb{S}_a(W)$ be a contact submanifold of (M, ξ) with fixed trivial normal bundle $S \times B^{2k}(R)$. Let $b \in \mathbb{N}$ be such that $R^2 \geq a/b$. The (a, b)–contact blow–up $\widetilde{M}_S$ of M along S is defined to be the contact cut of M for the moment map associated to the circle action $\varphi_{(a,-b)}$:

$$\widetilde{M}_S := M_{\{\mu_{(a,b)} \leq 0\}}$$

[2]The embedding $e : (M_1, \xi_1) \longrightarrow (M_2, \xi_2)$ is isocontact if $e^*(\xi_2) = \xi_1$.

The collapsed region $\mu^{-1}_{(a,b)}(0)/\sim$ will be called the *exceptional divisor*, it is a contact manifold of dimension $2n-1$. The induced $\mathbb{S}^1$–action on the level set

$$\mu^{-1}_{(a,b)}(0) \cong S \times \{\sqrt{a/b}\} \times \mathbb{S}^{2k-1}$$

coincides with the action $\varphi_{(a,-b)}$ defined in Theorem 3.1 with $W_1 = W$ and $W_2 = \mathbb{CP}^{k-1}$. Thus, the orbit space is

$$\mu^{-1}_{(a,b)}(0)/\mathbb{S}^1 \cong \mathbb{S}_{(b,a)}(W \times \mathbb{CP}^{k-1}) \cong \mathbb{S}(W) \times \mathbb{S}(\mathbb{CP}^{k-1})/\sim .$$

Remark 6.4. Notice that both the topology and the contact structure of the exceptional divisor strongly depend on the choice of the parameters a and b. Consequently, so does $\widetilde{M}_S$.

Example: 1. In the case of a contact 5–fold, a transverse circle –the simplest contact submanifold– is replaced by a (quotient of a) standard contact 3–sphere, as in Section 4. This new construction of the blow–up along a transverse loop will be compared with the previous ones in the next section.

2. Consider the contact blow–up along a contact 3–sphere $\mathbb{S}^3 \cong \mathbb{S}(\mathbb{CP}^1) \subset M^{2n+1}$. The normal bundle is necessarily trivial because $\pi_2(SO(n)) = \{0\}$. The exceptional divisor will be $\mathbb{S}_{(b,a)}(\mathbb{CP}^1 \times \mathbb{CP}^{n-2})$. Confer Example 3 in Section 3.

3. In the previous example, suppose that (M,ξ) is a 5–dimensional contact manifold. Then the exceptional divisor of the $(1,k)$ blow–up is contactomorphic to $\mathbb{S}^3$. In higher dimensions, the exceptional divisor of a $(1,k)$ blow–up along $\mathbb{S}^3$ is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^{2n-3}$ for $n \geq 3$ and k even.

6.3. Blow–up for general normal bundle. We define the contact blow–up along a contact submanifold with a general normal bundle. The construction will clearly coincide with the previous blow–up in the case of a trivial normal bundle.

6.3.1. *Preliminaries.* In smooth topology the smooth structure of a neighbourhood of a submanifold is retained by the normal bundle. The contact geometry nearby a contact submanifold (S,ξ_S) is determined by the symplectic structure on the normal bundle ν_S. Such a structure exists because ν_S can be identified with the symplectic orthogonal $\xi_S^{\perp}$. The contact neighbourhood theorem is as follows:

THEOREM 6.2. (*2.5.15 in* [**61**]) *Let* (S_1, M_1) *and* (S_2, M_2) *be contact pairs such that* (S_1, ξ_{S_1}) *is contactomorphic to* (S_2, ξ_{S_1})*. If* $\xi_{S_1}^{\perp} \cong \xi_{S_2}^{\perp}$ *as conformally symplectic vector bundles, then there exists a contactomorphism between suitable neighbourhoods of* S_1 *and* S_2*.*

There exist contact submanifolds with non–trivial normal bundle in a closed contact manifold. Let us provide some examples.

Examples: 1. Let $(M, \xi = \ker \alpha)$ be a cooriented contact manifold and ξ itself be non–trivial as an abstract vector bundle. The contact form provides a contact embedding $\alpha : M \longrightarrow \mathbb{S}(T^*M)$ such that the normal bundle of the contact submanifold M is isomorphic to ξ.

2. Let (M^{2n+1}, ξ) be a closed cooriented contact manifold. Consider an isocontact embedding

$$(M^{2n+1}, \xi) \longrightarrow (\mathbb{S}^{4n+3}, \xi_{std}),$$

see [**73**] for the existence of such an embedding. Since the tangent bundle of the spheres are stably trivial it is simple to give sufficient conditions for the normal bundle to be non–trivial, e.g. M not spin.

REMARK 6.5. The contact blow–up construction has been used in another context. Given a complex vector bundle E on M, the contact submanifold $S \subset M$ is defined as the vanishing set of a section in $H^0(M, E)$. Then $c_1(\nu(S)) = PD([S]) \neq 0$. This occurs for the base locus of contact Lefschetz pencil decompositions of (M, ξ), a situation encountered in Chapter 2.

6.3.2. *Definition.* In the blow–up construction for the trivial normal bundle case there are two circle actions. The first one exists on the contact submanifold S, since it is a Boothby–Wang manifold, and it is extended to a local neighbourhood. The second circle action is the gauge action provided by the complex structure on the conformally symplectic normal bundle. While this second action is still available in the non–trivial normal bundle case, the first action can *a priori* no longer be extended to a neighbourhood.

We hence require a lifting condition for the circle action on S: the appropriate set–up is as in Diagram 1 in Section 5:

$$\begin{tikzcd}
\nu_M(S) \cong \pi^*(V) \arrow[d] & \mathbb{S}(V) \arrow[d] \\
(S,\xi_S) \cong \mathbb{S}_a(W) \arrow[d, "\pi"'] & (V,\overline{\omega}) \arrow[dl] \\
(W,\omega) &
\end{tikzcd}$$

where V is a symplectic vector bundle over a symplectic manifold W. Assume $a = 1$ for simplicity.

Lemma 6.6. *Under the hypotheses above, the circle action provided by the Boothby–Wang structure can be naturally extended to a neighbourhood of S.*

Proof. Since W is a symplectically embedded submanifold of V, $\mathbb{S}(W)$ is a contact submanifold of $\mathbb{S}(V)$. The tubular neighbourhood theorem tells us that the normal bundle $\nu_M(S)$ is diffeomorphic to a small neighbourhood of S in M, but after the smooth isomorphism $\nu_M(S) \cong \pi^*(V)$ the same situation applies to $\mathbb{S}(W)$ in $\mathbb{S}(V)$. Since the isomorphism holds at the level of symplectic bundles, the contact tubular neighbourhood theorem ensures that there exists a contactomorphism Φ between a contact neighbourhood of the zero section of $\nu_M(S)$ and a contact neighbourhood of $\mathbb{S}(W)$ in $\mathbb{S}(V)$. Consequently, the circle action on $\mathbb{S}(V)$ can be carried along Φ to a neighbourhood of S. □

Let us spell out the moment map of the circle action. We refer to the circle action on the normal bundle induced by its complex structure as the gauge action. This action is the natural $\mathbb{S}^1$–action when working with a contact pair (S, M). Further, the radius $r > 0$ is a global coordinate regardless of the non–triviality of the normal bundle. We shall refer to the other action described above as the Boothby–Wang action. It is the natural action when identifying a neighbourhood of S in M with a neighbourhood of $\mathbb{S}(W)$ in $\mathbb{S}(V)$ via the map Φ in the proof of Lemma 6.6.

Lemma 6.7. *The moment map of the $\mathbb{S}^1$–action $\varphi_{(1,-1)}$ is $1 - \Phi(r)^2$.*

Proof. The moment map of the gauge action is $-r^2$. For the Boothby–Wang action, the circle action realizes the Reeb vector field and thus its moment map is 1. We express the r coordinate through the contactomorphism Φ as $\Phi(r)$. Since we are using the action $\varphi_{(1,-1)}$ the statement follows. $\square$

Recall that the contact cut can be performed if 0 lies in the image of the moment map.

REMARK 6.8. The same argument using a multiple of the gauge action yields that we may modify the action in order to ensure the following: Φ maps the zero section to $\mathbb{S}(W)$ and thus the values of $1 - b^2\Phi(r)^2$ form a decreasing sequence in b that eventually crosses zero.

The Boothby–Wang action may as well be arranged to period a: the concatenation action is denoted $\varphi_{(a,-b)}$. We are in position to make the

DEFINITION 6.9. (Contact Blow–Up) Let $S \cong \mathbb{S}_a(W)$ be a contact submanifold of (M, ξ). Let $a, b \in \mathbb{Z}^+$ be such that the origin is contained in the image of the moment map $\mu_{(a,b)}$ for the action $\varphi_{(a,-b)}$. The (a,b)–contact blow–up $\widetilde{M}_S$ of M along S is defined to be the contact cut of M for the action $\varphi_{(a,-b)}$, i.e. $\widetilde{M}_S := M_{\{\mu_{(a,b)} \leq 0\}}$.

7. Uniqueness for Transverse Loops

In this section we relate the three constructions of the contact blow–up given in Sections 4, 5 and 6. The construction that can be performed in the most general situation is the one involving the contact cut. It has two degrees of freedom: a pair of positive integers a and b. These two parameters relate to previous integers appearing in the first two constructions. Indeed, the parameter l in the contact surgery blow–up corresponds to b. For Gromov's construction, the choice of collapsing radius $k \in \mathbb{N}$ gives rise, in the case of transverse loops, to the exceptional divisor $L(k, 1)$ and it corresponds to the parameter a. It is quite obvious that the diffeomorphism type of the blown–up manifolds is the same regardless of the chosen construction as soon as the parameters coincide as just mentioned.

Let us turn our attention to the contact structure: we restrict ourselves to the case of transverse loops. Denote by $\overline{M}_b$ the surgery contact blow–up defined in Section 4 with parameter b. The contact blow–up as defined in Section 5 with radius a is denoted by M'_a. And $\widetilde{M}_{(a,b)}$ will be the contact–cut blow–up as defined in Section 6, performed with parameters (a,b). Let us show that uniqueness holds in this case, more precisely we prove the following

THEOREM 7.1. *Let (M,ξ) be a contact manifold. Performing the blow–up along a fixed transverse loop with the three procedures introduced previously, the resulting blown–up manifolds $\overline{M}_1$, M'_1 and $\widetilde{M}_{(1,1)}$ endowed with the blown–up contact structures are contactomorphic. Further, given any pair of integers (a,b), the following contactomorphisms hold:*

$$(\overline{M}_b,\overline{\xi}_b) \cong \left(\widetilde{M}_{(1,b)},\widetilde{\xi}_{(1,b)}\right), \quad (M'_a,\xi'_a) \cong \left(\widetilde{M}_{(a,1)},\widetilde{\xi}_{(a,1)}\right).$$

The relation between the different constructions is already hinted in Section 4. Since the exceptional contact divisors coincide and the procedure is of local nature, i.e. the contact manifold is not altered away from a neighbourhood of the embedded transverse loop, our study will focus on the natural annulus contact fibration.

REMARK 7.1. In the three constructions a common trivialized neighborhood is fixed. Theorem 7.1 shows that the contact blow–up is unique up to the choice of a trivializing chart on the neighbourhood of the transverse loop. The space of isocontact embeddings of the contact manifold $\mathbb{S}^1 \times B^{2n}$ into M should be studied in order to prove the uniqueness of the blow–up along isotopy classes of transverse loops. It is probably false that this space is connected, which is needed to ensure the uniqueness of the blow–up once the parameters a,b are fixed.

Let us review a few facts.

A contact fibration is a fibration $(M,\xi) \longrightarrow B$ such that the fibers are contact submanifolds. We consider contact fibrations over the disk $f:(M,\xi) \longrightarrow B^2$. The base being contractible, the fibration is trivial and we also assume it to be trivialized. Let us introduce the following

DEFINITION 7.2. Let (r,θ) be polar coordinates on the disk B^2. A trivialized contact fibration over the disk $\pi: F \times B^2 \longrightarrow B^2$ is said to be radial if the contact structure admits the following equation

$$\ker \alpha_0 = \ker\{\alpha_F + H d\theta\}, \tag{7.1}$$

where $H: F \times B^2 \longrightarrow \mathbb{R}$ is a smooth function such that $H = O(r^2)$.

Notice that for the total space of a radial contact fibration to have an induced contact structure it is necessary that

$$\partial_r H > 0 \text{ for } r > 0. \tag{7.2}$$

It is convenient to adapt the previous definition to include the situation in which lens spaces appear as exceptional contact divisors:

Definition 7.3. A trivialized radial contact fibration $\pi : \mathbb{S}^{2n-1} \times B^2 \longrightarrow B^2$ is $\mathbb{Z}_a$–equivariant if the natural diagonal $\mathbb{Z}_a$–action on the fibration preserves the radial contact structure.

The action on the fiber sphere $\mathbb{S}^{2n-1}$ is generated by a $\frac{2\pi}{a}$–rotation along the Hopf fiber, whereas the action on the base B^2 is the standard $\frac{2\pi}{a}$–rotation of the disk. They preserve the standard contact structure on $\mathbb{S}^{2n-1}$ and the 1–form $d\theta$ on the disk, respectively. Hence, the fibration becomes equivariant if the function H is preserved by the action.

Topologically it is fairly straightforward that the blow–up operations we are performing are tantamount to a priori different fillings of the fibration over an annulus to form a manifold lying over the disk – this being always considered up to a finite $\mathbb{Z}_a$–action, for lens space fillings. The transition from $\mathbb{S}^1 \times B^{2n}$ to $B^2 \times \mathbb{S}^{2n-1}$ can be understood in the following way: both fibrations over the annulus –obtained by restricting to $r \in (1/2, 1)$– are filled in the origin with a circle and $\mathbb{S}^{2n-1}$ respectively. In the transverse loop case it will be enough to use the following

Lemma 7.4. *Let M be a manifold with contact structures ξ_0 and ξ_1. Assume that there are two smoothly isotopic diffeomorphisms*

$$f_0, f_1 : M \longrightarrow F \times B^2,$$

which are contactomorphisms[3] *for ξ_0 and ξ_1 respectively. Let the two fibrations be radial contact fibrations with common contact fiber F and such that the diffeomorphism*

$$f_1 \circ f_0^{-1} : F \times B^2 \longrightarrow F \times B^2$$

is the identity close to the boundary. Then, the contact structures ξ_0 are ξ_1 are isotopic.

Further, if the fiber is $F \cong \mathbb{S}^{2n-1}$ and the contact fibrations are $\mathbb{Z}_a$–equivariant, the contact structures are isotopic through $\mathbb{Z}_a$–equivariant contactomorphisms.

[3] A priori, not necessarily contact isotopic.

PROOF. This can be reduced to the setup of a fibration $F \times B^2$ with two different radial contact structures

$$\begin{aligned} \alpha_0 &= \alpha_F + H_0 d\theta, \\ \alpha_1 &= \alpha_F + H_1 d\theta, \end{aligned}$$

such that the Hamiltonians H_0 and H_1 coincide near to the boundary. In this setting, we just need to construct a path of functions $H_t : F \times B^2 \longrightarrow \mathbb{R}$ connecting them, relative to the boundary, satisfying the contact equation (7.2) and the condition $H_t = O(r^2)$. But this is possible since the space of such functions is convex.

The argument still works in the equivariant case: the only sentence to be added is that the space of equivariant Hamiltonians is also convex. □

Thus, to conclude uniqueness we study the contact topology of the different blow–up constructions and ensure that the lemma applies.

Proof of Theorem 7.1: Let us describe the common model fibration that underlies the three constructions in this case. Consider a standard contact neighbourhood $\mathbb{S}^1 \times (0,2) \times \mathbb{S}^{2n-1}$ of the given fixed loop and the morphism

$$\begin{aligned} \phi_{(a,b)} : (\mathbb{S}^1 \times (0,2) \times \mathbb{S}^{2n-1}) &\longrightarrow \mathbb{S}^1 \times (0,2) \times \mathbb{S}^{2n-1} \\ (\theta, r, z) &\longrightarrow (a\theta, r, e^{2\pi i b\theta} z). \end{aligned}$$

It generalizes the diffeomorphism in equation (4.2) that corresponds to the case $a = 1$. If a is greater than 1, $\phi_{(a,b)}$ becomes an $a:1$ covering. The covering transformation is provided by $\mathbb{Z}_a$ acting by

$$\begin{aligned} \mathbb{Z}_a \times (\mathbb{S}^1 \times (0,2) \times \mathbb{S}^{2n-1}) &\longrightarrow (\mathbb{S}^1 \times (0,2) \times \mathbb{S}^{2n-1}) \\ (l, (\theta, r, p)) &\longrightarrow \left(\frac{2\pi l}{a} + \theta, r, e^{2\pi i b l/a} p \right). \end{aligned}$$

To understand the change in the contact structure, note that the pull–back of the standard contact form $\eta = d\theta - r^2 \alpha_{std}$ is given by

$$\lambda = \phi^*_{(a,b)} \eta = (-r^2) \cdot [(b - ar^{-2}) d\theta + \alpha_{std}].$$

Denote by $R_0 = \left(\frac{a}{b}\right)^{1/2}$ the critical radius where the distribution becomes horizontal. In order to have the blow–up procedure properly defined, enough radius is required for the tubular neighborhood of the trivialization. This corresponds to the condition $R_0 < 2$. In these coordinates, for any fixed small $\varepsilon > 0$, the projection onto the first two factors

$$\pi : \mathbb{S}^1 \times (R_0 + \varepsilon, 2) \times \mathbb{S}^{2n-1} \longrightarrow \mathbb{S}^1 \times (R_0 + \varepsilon, 2)$$

provides a radial contact fibration over the annulus, and since the function $(ar^{-2} - b)$ is strictly positive in $(R_0 + \varepsilon, 2)$, it can be extended to the interior of the disk to a $\mathbb{Z}_a$–equivariant radial contact fibration. In order to glue back the model to the manifold we should quotient the equivariant contact fibration by $\mathbb{Z}_a$, this allows us to use the map $\phi_{(a,b)}$ to insert the model back into the manifold.

It thus remains to verify that the three blow–up procedures provide examples of such an extension for particular values of (a, b). Then Lemma 7.4 will apply to provide the uniqueness of the constructions. Note that the contact surgery blow–up construction is by definition a radial contact fibration, with $a = 1$, as shown in Section 4. Let us study the two remaining cases.

To understand the construction in Section 5, let us proceed backwards and instead of applying the Boothby–Wang construction, we produce a contact structure and then quotient the resulting contact manifold by the Reeb $\mathbb{S}^1$–action to study whether it is the correct object. Once the coordinate change $\phi_{(a,b)}$ is performed, the Reeb vector field ∂_θ becomes

$$\phi^*_{(a,b)}(\partial_\theta) = \frac{1}{a}(\partial_\theta - bR_{std}).$$

This vector field extends to the interior of the disk fibration and so we may quotient the resulting manifold $B^2 \times \mathbb{S}^{2n-1}$. We obtain the blown–up symplectic ball $\widetilde{B}^{2n}$ as its quotient. We can further quotient by the free $\mathbb{Z}_a$–action to obtain a non–trivial fibration over the disk B^2. This proves that a suitable choice of connection leads to an equivariant contact fibration.

There are other choices of connection though. From the principal bundle point of view, a radial contact fibration over the annulus $\mathbb{S}^1 \times (0, 2)$ corresponds to a connection on

$$B^2 \times \mathbb{S}^{2n-1} \longrightarrow \widetilde{B}^{2n}.$$

Certainly, by Proposition 5.4 the contact structure is fixed by the choice of a connection. Note that the space of connections is affine and thus, by Gray stability, the resulting contact structures are contact isotopic for different choices of connections. In conclusion, this second model also provides an extension of the model fibration.

We describe the third procedure also beginning with the resulting contact manifold and giving the pull–back of the action. This contact cut construction is also an equivariant radial contact fibration since the pull–back of the vector field generating the $\mathbb{S}^1$–action, that is

$$X = a\partial_\theta - bR_{std},$$

is $\phi^*_{(a,b)}X = \partial_\theta$ after the coordinate change. Therefore, the contact cut is just an equivariant radial contact fibration, see the proof of Theorem 2.11 in [**89**]. $\square$

CHAPTER 5

On the strong orderability of overtwisted 3-folds

In this fifth chapter we address the existence of positive loops of contactomorphisms in overtwisted contact 3–folds, this is related to the previous chapters as explained in Chapter 1. We present a construction of such positive loops in the contact fibered connected sum of certain contact 3–folds along transverse knots. In particular, we obtain positive loops of contactomorphisms in a class of overtwisted contact structures. This is joint work with F. Presas.

1. Introduction

Let (M, ξ) be a connected contact manifold with a cooriented contact structure. In [**55**], Y. Eliashberg and L. Polterovich observed that the universal cover $\widetilde{\mathrm{Cont}}_0(M, \xi)$ of the identity component of the group of contactomorphisms carries a natural non–negative normal cone. This structure induces a partial binary relation on the groups $\widetilde{\mathrm{Cont}}_0(M, \xi)$ and $\mathrm{Cont}_0(M, \xi)$. This relation is naturally reflexive and transitive but not necessarily anti–symmetric. In case it is anti–symmetric it provides a partial order on these groups. This has been of central interest [**55, 53, 72**] in contact topology.

The existence of this partial order in $\widetilde{\mathrm{Cont}}_0(M, \xi)$ can be stated in terms of the non–existence of positive contractible loops of contactomorphisms, confer Section 2 below. In particular, this leads to the study of positive loops of contactomorphisms and that of positive Legendrian isotopies (see for instance [**38, 34, 35**]). A significant part of the current knowledge on the subject can be subsumed as follows. The contact jet spaces $J^1(\mathbb{R}^n)$ and $J^1(\mathbb{R}^n, \mathbb{S}^1)$ along with the spaces of cooriented contact elements do not admit a positive contractible loop of contactomorphisms [**14, 38, 35, 53, 116**]. The standard contact structure on a sphere $\mathbb{S}^{2n+1}$, different from $\mathbb{S}^1$, does admit a positive contractible loop of contactomorphisms [**53, 72, 110**].

The method used in [**35**] also implies that the space of contact elements of $\mathbb{T}^2$ is strongly orderable, that is, it does not even admit a positive loop of contactomorphisms. In general, [**35**, Corollary 9.1] implies that the cosphere bundle of a manifold with infinite fundamental group does not admit a positive loop of contactomorphisms. In [**2**, Theorem 7.1] the cosphere bundle of a manifold with finite fundamental group (and rational cohomology ring with at least two generators) is also shown to be strongly orderable. In this direction, P. Weigel [**121**] shows that the existence of a non–standard symplectic ball whose Rabinowitz Floer homology growth rate is superlinear can be used to locally perturb any higher–dimensional Liouville fillable contact structure to a strongly orderable contact structure.

The canonical contact structures on $J^1(\mathbb{R})$ and $J^1(\mathbb{R},\mathbb{S}^1)$ and those obtained as the space of cooriented contact elements of a surface are tight contact structures. Thus the list above does not include any overtwisted contact 3–fold. This Chapter presents the first examples of positive loops in overtwisted contact 3–folds. The first result towards the understanding of positive loops of contactomorphisms in overtwisted 3–folds appears in Chapter 6 of this dissertation. There, the non–existence of positive loops generated by a Hamiltonian with a small C^0–norm has been proven. This statement sided with the folklore conjecture that overtwisted contact manifolds do not admit positive loops of contactomorphisms. Surprisingly, Theorem 1.1 below provides a counter–example to the conjecture.

In the present Chapter, we prove that there exist overtwisted contact structures admitting positive loops of contactomorphisms. This is achieved with an explicit construction involving the fibered connected sum with $(\mathbb{S}^1\times\mathbb{S}^2,\xi_{st})$ along a transverse knot. Denote by (M,ξ^κ) the contact structure obtained from (M,ξ) by performing a half Lutz twist along a transverse knot κ. The main result we shall provide is the following

THEOREM 1.1. *Let (M,ξ) be a contact 3–fold that admits a positive loop of contactomorphims $\{\phi_t\}$. Suppose that there exists a locally autonomous orbit κ of $\{\phi_t\}$. Then the overtwisted contact 3–fold (M,ξ^κ) admits a positive loop of contactomorphisms.*

In conjuction with the results of Chapter 6, the Hamiltonians generating such positive loops cannot be C^0–small. That is, given a contact structure $(M,\ker\alpha)$ there exists a positive constant $C(\alpha)$ such that for any Hamiltonian $H : M \times \mathbb{S}^1 \longrightarrow \mathbb{R}$ generating a positive loop, we have $\|H\|_{C^0} \geq C(\alpha)$.

The notion of a locally autonomous orbit appearing in Theorem 1.1 is introduced in Section 3. For instance, the Boothby–Wang manifold associated to a surface conforms the hypothesis of Theorem 1.1.

COROLLARY 1.1. *Let* (Σ, ω) *be a symplectic 2–dimensional orbifold. The contact structure obtained by a half Lutz twist along a positive transverse regular fibre of the circle orbibundle* $\mathbb{S}(\Sigma, \omega)$ *admits a positive loop of contactomorphisms.*

This yields a positive loop of contactomorphisms for the overtwisted contact structures $(\mathbb{S}^3, \xi_k)$ corresponding to positive integers $k \in \mathbb{Z}^+$ representing the homotopy classes $k \in H^3(M, \pi_3(\mathbb{S}^2)) \cong \mathbb{Z}$.
Theorem 1.1 also applies to the (unique) tight contact structure ξ_{st} on $\mathbb{S}^1 \times \mathbb{S}^2$.

COROLLARY 1.2. *The contact structure* $(\mathbb{S}^1 \times \mathbb{S}^2, \xi^\kappa)$ *obtained by a half Lutz twist along the positive transverse knot* $\kappa = \mathbb{S}^1 \times \{(0, 0, 1)\} \subset (\mathbb{S}^1 \times \mathbb{S}^2, \xi_{st})$ *admits a positive loop of contactomorphisms.*

The existence of such positive loops implies squeezing phenomena on the aforementioned contact 3–folds. Nevertheless we cannot conclude its contractibility and thus the squeezing in the isotopy sense does not follow [**53**]. Similarly, Theorem 1.1 implies that the binary relation [**55**] is not a partial order in $\mathrm{Cont}_0(M, \xi)$ for these overtwisted manifolds, but the lift to the universal cover might still be a partial order. See Subsection 2.4 for details.

The Chapter is organized as follows. Section 2 contains the required preliminaries in contact topology. The construction used in order to prove Theorem 1.1 involves a fibered connected sum with $\mathbb{S}^1 \times \mathbb{S}^2$. Section 3 presents this contact manifold and describes a certain non–negative loop of contactomorphisms. In Section 4 we prove Theorem 1.1 using the tools in Section 2 and the loop in Section 3.

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2. Preliminaries

In this section we briefly introduce the basic ingredients involved in Theorem 1.1. Subsections 2.1, 2.2 and 2.3 can be essentially extracted from [**61**]. The reader is referred to [**53, 72**] for Subsection 2.4. In this Chapter (M,ξ) denotes a contact 3–fold.

2.1. Fibered connected sum. Let us consider the 3–fold

$$\mathbb{S}^1 \times D^2(R) = \{(\theta; x, y) : x^2 + y^2 \leq R\} = \{(\theta; r, \varphi) : r \leq R\}$$

with the contact structure ξ_0 defined by the contact form $\alpha_0 = d\theta + r^2 d\varphi$.

Suppose that $\gamma : \mathbb{S}^1 \longrightarrow (M,\xi)$ is a transverse knot with a fixed frame $\tau : \mathbb{S}^1 \longrightarrow \gamma^*\xi$. Then, for $R > 0$ small enough, there exists a unique (up to contact isotopy) contact embedding

$$\phi : (\mathbb{S}^1 \times D^2(R), \xi_0) \longrightarrow (\phi(\mathbb{S}^1 \times D^2(R)), \xi) \subset (M, \xi)$$

such that $\phi(\theta, 0, 0) = \gamma(\theta)$ and the frame $\phi^*\tau : \mathbb{S}^1 \longrightarrow \xi_0$ is homotopic to ∂_x.

Given two framed transverse knots (γ_1, τ_1) and (γ_2, τ_2) in two contact 3–folds (M_1, ξ_1) and (M_2, ξ_2), we can define the fibered connected sum along these knots. It is described as follows.
Consider the domain $A_R = \mathbb{S}^1 \times (-R^2, R^2) \times \mathbb{S}^1$ with coordinates (θ, v, φ) and the contact form $\eta = d\theta + v d\varphi$. Then the pair of gluing maps:

$$\begin{aligned} g_1 : \mathbb{S}^1 \times (D^2(R) \setminus \{0\}) &\longrightarrow \mathbb{S}^1 \times (0, R^2) \times \mathbb{S}^1 \subset A_R \\ (\theta, r, \varphi) &\longmapsto (\theta, r^2, \varphi) \end{aligned}$$

$$\begin{aligned} g_2 : \mathbb{S}^1 \times (D^2(R) \setminus \{0\}) &\longrightarrow \mathbb{S}^1 \times (-R^2, 0) \times \mathbb{S}^1 \subset A_R \\ (\theta, r, \varphi) &\longmapsto (\theta, -r^2, -\varphi) \end{aligned}$$

satisfy $g_1^*\eta = g_2^*\eta = \alpha_0$ and thus are strict contact embeddings. Then the contact fibered connected sum along (γ_1, τ_1) and (γ_2, τ_2) is the smooth manifold

$$(M_1, \xi_1)\#(M_2, \xi_2) := (M_1 \setminus \gamma_1(\mathbb{S}^1)) \cup_{g_1 \circ \phi_1^{-1}} A_R \cup_{g_2 \circ \phi_2^{-1}} (M_2 \setminus \gamma_2(\mathbb{S}^1))$$

where ϕ_1 and ϕ_2 are the contact embeddings corresponding to (γ_1, τ_1) and (γ_2, τ_2). This 3–fold is endowed with a contact structure in each piece and, since these are glued with g_1 and g_2, there exists a contact structure on $(M_1, \xi_1)\#(M_2, \xi_2)$.

Observe that an isotopy of framed transverse knots preserves the isotopy class of the resulting contact structure (by Gray's stability). Also, the isotopy class of the contact structure does not depend on each of the frames (τ_1, τ_2) but only on their sum:

LEMMA 2.1. *The contact structure on the fibered sum* $(M_1, \xi_1)\#(M_2, \xi_2)$ *along* (γ_1, τ_1) *and* (γ_2, τ_2) *is isotopic to the contact structure on the fibered connected sum* $(M_1, \xi_1)\#(M_2, \xi_2)$ *along* $(\gamma_1, \tau_1 + k)$ *and* $(\gamma_2, \tau_2 - k)$ *for any* $k \in \mathbb{Z}$.

The fibered connected sum along a framed transverse knot can be used to modify the contact structure of a 3–fold (while preserving its diffeomorphism type). Indeed, the connected sum $M\#(\mathbb{S}^1 \times \mathbb{S}^2)$ along the knot $\mathbb{S}^1 \times \{pt.\}$ is diffeomorphic to M. This operation yields a non–trivial operation from the contact topology viewpoint, the half Lutz twist.

2.2. The half Lutz twist. Consider $\mathbb{S}^1 \times \mathbb{R}^3$ with coordinates given by $(\theta; x, y, z) \simeq (\theta; r, \varphi, z)$ and the contact manifold

$$(\mathbb{S}^1 \times \mathbb{S}^2, \xi_{st}) = (\{(\theta; r, \varphi, z) : r^2 + z^2 = 1\}, \ker\{z d\theta + r^2 d\varphi\}) \subset \mathbb{S}^1 \times \mathbb{R}^3.$$

This is the unique tight contact structure on $\mathbb{S}^1 \times \mathbb{S}^2$, see [**70**].

Let (Γ, ι) be the framed transverse knot on $\mathbb{S}^1 \times \mathbb{S}^2$ defined by $\Gamma(\theta) = (\theta; 0, 0, 1)$ and $\iota(\theta) = \partial_x$.

DEFINITION 2.2. Let (M, ξ) be a contact 3–fold and (γ, τ) a framed transverse knot. The half Lutz twist of (M, ξ) along the transverse knot γ is the contact fibered connected sum $(M, \xi)\#(\mathbb{S}^1 \times \mathbb{S}^2, \xi_{st})$ along (γ, τ) and (Γ, ι).

The half Lutz twist of (M, ξ) along a transverse knot γ is denoted by (M, ξ^γ). Note that the action of $\Omega SO(3) \subset \mathrm{Diff}(\mathbb{S}^1 \times \mathbb{S}^2)$ implies that the diffeomorphism type of $(M, \xi)\#(\mathbb{S}^1 \times \mathbb{S}^2, \xi_{st})$ is independent of the choice of frame ι and thus equal to M. In terms of surgeries, it is a Dehn surgery in which the meridian is sent to the meridian and thus the smooth type of the resulting manifold remains the same. Similarly, the contact structure ξ^γ does not depend either on the choice of frame, see [**45, 61**].

There are two relevant features regarding (M, ξ^γ). First, it is an overtwisted contact 3–fold. There is a family of overtwisted disks that appear from the family of immersed overtwisted disks $\{\theta\} \times \mathbb{S}^2$ in the tight $(\mathbb{S}^1 \times \mathbb{S}^2, \xi_{st})$, whose boundaries (collapsed at a point) form the knot Γ.

Second, the homotopy class of ξ^γ differs from that of ξ. The primary obstruction is the class

$$d^2(\xi,\xi^\gamma) = c_1(\xi) - c_1(\xi^\gamma) = -2PD([\gamma]) \in H^2(M,\pi_2(\mathbb{S}^2)).$$

The positive loop of contactomorphims obtained in Theorem 1.1 is essentially built separately in the two pieces of a fibered connected sum. The first piece is the given contact 3–fold (M,ξ) and the other corresponds to $(\mathbb{S}^1\times\mathbb{S}^2,\xi_{st})$. The loop is constructed by gluing a positive loop in each of the pieces, thus resulting in a positive loop of contactomorphims for the half Lutz twist of (M,ξ).

2.3. Loops of contactomorphisms. Let (M,ξ) be a contact structure and a 1–form α such that $\xi=\ker\alpha$. The choice of α uniquely determines a vector field R_α such that

$$i_{R_\alpha}\alpha = 1, \quad i_{R_\alpha}d\alpha = 0.$$

A vector field X is said to be contact if $\mathcal{L}_X\alpha = f\alpha$, for some $f\in C^\infty(M)$. Given a contact vector field X, the function $H=\alpha(X)\in C^\infty(M)$ satisfies the equations

$$\begin{aligned} i_X\alpha &= H, \\ i_X d\alpha &= (d_{R_\alpha}H)\alpha - dH. \end{aligned}$$

Conversely, given a function $H\in C^\infty(M)$ there exists a unique contact vector field X verifying the equations above. The function H is called the Hamiltonian function associated to X. This establishes a linear isomorphism (depending on α) between the vector space of contact vector fields and the vector space of smooth functions.

The correspondence can be made time–dependent. Given a time–dependent flow $\phi_t : M\times[0,1]\longrightarrow M$ of contactomorphisms, its associated time–dependent vector field is defined by

$$\dot{\phi}_t = X_t\circ\phi_t.$$

The function $H_t=\alpha(X_t) : M\times[0,1]\longrightarrow\mathbb{R}$ will be referred to as the Hamiltonian generating the contact flow ϕ_t, and denoted by $H(\phi_t)$. It will we assumed to be 1–periodic in time. The flow of contactomorphisms ϕ_t is said to be a smooth loop if $\phi_1 = id$ and the quotient map $\phi_t : M\times\mathbb{S}^1\longrightarrow M$ is smooth. The loop of contactomorphisms is positive if its generating Hamiltonian is positive, i.e. $H_t(p,t)>0$ at any $(p,t)\in M\times\mathbb{S}^1$.

There are two useful operations in the spaces of loops of contactomorphisms: concatenation and composition. The concatenation is defined as follows. Let $\{\Phi_t^1, \dots, \Phi_s^l\}$ be a set of $l \in \mathbb{Z}^+$ loops of contactomorphisms respectively generated by Hamiltonians $\{F_t^1, \dots, F_t^l\}$.

The concatenation of the loops $\{\Phi_t^1, \dots, \Phi_t^l\}$ is defined as

$$\Phi_t^1 \odot \dots \odot \Phi_t^l = \begin{cases} \Phi_{lt}^1 & t \in [0, 1/l], \\ \Phi_{lt-1}^2 & t \in [1/l, 2/l], \\ & \vdots \\ \Phi_{lt-l+2}^{l-1} & t \in [1-2/l, 1-1/l], \\ \Phi_{lt-l+1}^l & t \in [1-1/l, 1]. \end{cases}$$

The generating Hamiltonian $C : M \times \mathbb{S}^1 \longrightarrow \mathbb{R}$ for the concatenation is

$$C_t = H(\Phi_t^1 \odot \dots \odot \Phi_t^l) = \begin{cases} F^1(\cdot, lt) & t \in [0, 1/l], \\ F^2(\cdot, lt-1) & t \in [1/l, 2/l], \\ & \vdots \\ F^{l-1}(\cdot, lt-l+2) & t \in [1-2/l, 1-1/l], \\ F^l(\cdot, lt-l+1) & t \in [1-1/l, 1]. \end{cases}$$

Let Φ_t and Ψ_t be two loops of contactomorphisms generated by F_t and G_t. The second operation is the composition $\{\Phi_t \circ \Psi_t\}_t$ of Φ_t and Ψ_t. Suppose that the first loop satisfies $\Phi_t^* \alpha = e^{f_t} \alpha$, then the Hamiltonian generating the composition is

$$H(\Phi_t \circ \Psi_t)(p, t) = F_t(p, t) + e^{-f_t} G_t(\Phi_t^{-1}(p), t).$$

In addition, the conjugation $\{\psi \circ \Phi_t \circ \psi^{-1}\}_t$ of the loop Φ_t by a contactomorphism $\psi \in \mathrm{Cont}(M, \xi)$, such that $\psi^* \alpha = e^f \alpha$, is a loop of contactomorphisms generated by the Hamiltonian

$$H(\psi \circ \Phi_t \circ \psi^{-1})(p, t) = e^{-f} F_t(\psi^{-1}(p), t).$$

These operations will be used in Section 4.

2.4. Orderability. Let us consider the identity component of the group of contactomorphisms $G = \mathrm{Cont}_0(M, \xi)$ and its universal cover $\widetilde{G} = \widetilde{\mathrm{Cont}}_0(M, \xi)$. These groups are endowed with a natural relation. Given $f, g \in \mathrm{Cont}_0(M, \xi)$, the relation is defined as $f \geq g$ if and only if there exists a path ϕ_t of contactomorphisms such that $\phi_1 = f \circ g^{-1}$ and its generating Hamiltonian is non–negative. This relation is reflexive and transitive. Similarly, given two elements $[\phi_t], [\psi_t] \in \widetilde{G}$. The relation $[\phi_t] \geq [\psi_t]$ if and only if $[\phi_t \circ \psi_t^{-1}]$ admits a representative generated by a non–negative Hamiltonian is reflexive and transitive.

The contact manifold (M, ξ) is said to be strongly orderable if the relation $(G, \geq)$ is antisymmetric (and thus defines a genuine partial order). It is said to be orderable if the relation $(\widetilde{G}, \geq)$ is also antisymmetric. The following criterion relates the existence of this genuine partial order with the existence of positive loops of contactomorphisms:

PROPOSITION 2.3. [**55**, Criterion 1.2.C] *The relation $\geq$ is a non–trivial partial order on G if and only if there are no loops of contactomorphisms of (M, ξ) generated by a strictly positive Hamiltonian.*

In addition, the relation $\geq$ is a non–trivial partial order on $\widetilde{G}$ if and only if there are no contractible loops of contactomorphisms of (M, ξ) generated by a strictly positive Hamiltonian.

Theorem 1.1 implies the existence of non–strongly orderable overtwisted contact 3–folds. This is the first result relating overtwisted 3–folds to orderability.

3. Locally autonomous loops of contactomorphisms

Let (M, ξ) be a contact 3–fold and α an associated contact form. The fibered connected sum along a tranverse knot has been described in Subsection 2.1. The aim of this section is to introduce a property for a positive loop of contactomorphisms that allows us to obtain a positive loop of contactomorphisms in the fibered connected sum $(M, \xi)\#(\mathbb{S}^1 \times \mathbb{S}^2, \xi_{st})$.

This appropriate class of loops are the locally autonomous loops, described as follows. Let $p \in M$ be a point and $\{\phi_t\}$ a positive loop of contactomorphisms generated by a Hamiltonian F_t.

DEFINITION 3.1. The loop $\{\phi_t\}$ is said to be locally autonomous at p if there exists $\mathrm{Op}(p)$ such that

$$F(\phi_t(q), t_0) = F(\phi_t(q), t_1), \quad \forall t, t_0, t_1 \in \mathbb{S}^1 \text{ and } \forall q \in \mathrm{Op}(p)$$

and the map $\phi_t(p) : \mathbb{S}^1 \longrightarrow M$ is an embedding.

Observe that this definition does not depend on the choice of contact form α for ξ. The local autonomy at p is equivalent to $\dot{\phi}_t$ being time–independent on the trajectories passing through $\mathrm{Op}(p)$ and a positive loop that is locally autonomous at any point of the manifold is time–independent.

There exists also a normal form in a neighborhood of the orbit of the point p. It is used in order to glue the dynamics in a fibered connected sum. The normal form is the content of the following

PROPOSITION 3.2. *Let $\{\phi_t\}$ be a locally autonomous loop around p. Then there exist a constant $\rho \in \mathbb{R}^+$, a tubular neighborhood T_p of the orbit through p and a contactomorphism*

$$\psi : (\mathbb{S}^1 \times D^2(\rho), \ker\{\alpha_0 = d\theta + r^2 d\varphi\}) \longrightarrow (T_p, \xi|_{T_p}) \text{ such that } \alpha_0(\psi^* \dot{\phi}_t) = 1.$$

PROOF. Consider the contact form $\eta = \alpha/F_0$. The contact Hamiltonian H_t associated to the loop $\{\phi_t\}$ with respect to η satisfies $H_0 = 1$, and thus the contact vector field X_t coincides with the Reeb field R_η at $t = 0$. The strict Darboux Theorem ([**61**, Section 2.5]) implies the existence of a constant $\rho \in \mathbb{R}^+$, a neighborhood U_p and a strict contactomorphism

$$f : ((-\varepsilon, \varepsilon) \times D^{2n}(\rho), \alpha_0) \longrightarrow (U_p, \eta).$$

We can suppose that the neighborhood U_p is contained in the neighborhood $\mathrm{Op}(p)$ provided by Definition 3.1. Since the Reeb flow is a strict contactomorphism and the flow ϕ_t is locally autonomous on $\mathrm{Op}(p)$, the flow ϕ_t coincides with the Reeb flow in $\mathrm{Op}(p)$, and hence it is a strict contact flow.

The positive loop Ψ_t generated by the Reeb field on $(\mathbb{S}^1 \times D^{2n}(\rho), \ker\{\alpha_0\})$ is $\Psi_t(\theta, x) = (\theta + t, x)$. The neighborhood T_p is obtained through the flow of U_p and the contactomorphism ψ is the strict contact embedding

$$\begin{array}{rcl} \psi : \mathbb{S}^1 \times D^{2n}(\rho) & \longrightarrow & M \\ (\theta, x) & \longmapsto & \phi_\theta(f(\Psi_{-\theta}(\theta, x))). \end{array}$$

□

Consider two contact 3–folds (M, ξ) and (N, η) with positive loops of contactomorphisms Φ_t and Ψ_t locally autonomous at $p \in M$ and $q \in N$. Let γ and κ be the orbits of p and q with respect to Φ_t and Ψ_t, which come equipped with natural framings provided by Proposition 3.2. The fibered connected sum along γ and κ, introduced in Subsection 2.1, admits a positive loop of contactomorphisms.

This positive loop is defined as Φ_t and Ψ_t in $M \setminus \mathrm{Op}(\gamma)$ and $N \setminus \mathrm{Op}(\kappa)$, considered as submanifolds of $(M,\xi)\#(N,\eta)$ and extended to the gluing region of $(M,\xi)\#(N,\eta)$ with each of the two loops of contactomorphisms. In detail, Proposition 3.2 provides a normal form for both neighborhoods $\mathrm{Op}(\gamma)$ and $\mathrm{Op}(\kappa)$. This allows us to glue the two corresponding Hamiltonians $H(\Phi_t)$ and $H(\Psi_t)$ in their local normal form, both being constant on the gluing region and thus coinciding at $\mathbb{S}^1 \times \{0\} \times \mathbb{S}^1 \subset A_R$. This positive loop of contactomorphisms of $(M,\xi)\#(N,\eta)$ is denoted by $\Phi_t \# \Psi_t$.

The proof of Theorem 1.1 consists of this construction applied to the manifold $(\mathbb{S}^1 \times \mathbb{S}^2, \xi_{st})$ with an appropriate loop of contactomorphisms. The overtwistedness of the resulting contact structure follows from Subsection 2.2. Section 4 provides this loop and concludes Theorem 1.1.

4. Proof of the main result

In this section Propositions 4.1 and 4.2 are used to prove Theorem 1.1. Consider coordinates $(\theta; r, \varphi, z) \in \mathbb{S}^1 \times \mathbb{R}^3$ and the contact form $\alpha_{st} = zd\theta + r^2 d\varphi$ on the manifold $\mathbb{S}^1 \times \mathbb{S}^2 = \{(\theta; r, \varphi, z) : r^2 + z^2 = 1\} \subset \mathbb{S}^1 \times \mathbb{R}^3$. We can define the two solid tori

$$\mathbb{T}_1 = \mathbb{S}^1 \times \mathbb{D}^2 = \{(\theta; r, \varphi, z) : r^2 + z^2 = 1, z \geq 0\},$$
$$\mathbb{T}_2 = \mathbb{S}^1 \times \mathbb{D}^2 = \{(\theta; r, \varphi, z) : r^2 + z^2 = 1, z \leq 0\}.$$

There exists a non–negative autonomous Hamiltonian $R_t : \mathbb{S}^1 \times \mathbb{S}^2 \longrightarrow \mathbb{R}$ defined as $R_t(\theta; r, \varphi, z) = r^2$ which generates the non–negative loop of contactomorphisms $\{\rho_t\}$ given by

$$\rho_t(\theta; r, \varphi, z) = (\theta; r, \varphi + t, z).$$

A second autonomous Hamiltonian is also central to our construction. It is the Hamiltonian $Z_t : \mathbb{S}^1 \times \mathbb{S}^2 \longrightarrow \mathbb{R}$ defined as $Z_t(\theta; r, \varphi, z) = z$ whose associated loop of contactomorphisms $\{\zeta_t\}$ is

$$\zeta_t(\theta; r, \varphi, z) = (\theta + t; r, \varphi, z).$$

It is a loop of strict contactomorphisms, i.e. $\zeta_t^* \alpha_{st} = \alpha_{st}$.

The loops ρ_t and ζ_t are autonomous and commute, however only ρ_t is non–negative. Let us construct a locally autonomous positive loop of contactomorphisms in $(\mathbb{S}^1 \times \mathbb{S}^2, \xi_{st})$. It is obtained in two steps corresponding to the two subsequent Propositions.

PROPOSITION 4.1. *There exists a loop* $\{\beta_t\} \in \Omega\,\mathrm{Cont}(\mathbb{S}^1 \times \mathbb{S}^2, \xi_{st})$ *which coincides with* $\{\rho_t \circ \rho_t\}$ *on the solid torus* $\mathbb{T}_1$ *and it is positive on the solid torus* $\mathbb{T}_2$.

The proof follows closely the argument of [**55**, Prop. 2.1.B] and [**72**, Prop. 2.3].

PROOF. Consider the transverse knot $\gamma(\theta) = (-\theta; 0, 0, -1)$ in $(\mathbb{S}^1 \times \mathbb{S}^2, \xi_{st})$. Suppose that for a small enough neighborhood $\mathrm{Op}(\gamma)$ there exists a contactomorphism $\psi \in \mathrm{Cont}(\mathbb{S}^1 \times \mathbb{S}^2, \xi_{st})$ supported in $\mathrm{Op}(\gamma)$ and such that $\gamma \cap \psi(\gamma) = \emptyset$. Then the loop $\beta_t = \rho_t \circ \psi \circ \rho_t \circ \psi^{-1}$ coincides with $\rho_t \circ \rho_t$ on $\mathbb{T}_1$ and its Hamiltonian

$$H(\beta_t) = R_t(p,t) + H(\psi \circ \rho_t \circ \psi^{-1})(\rho_t^{-1}(p), t) = R_t(p,t) + e^{-f} R_t((\rho_t \circ \psi)^{-1}(p), t)$$

is positive on $\mathbb{T}_2$ since at least one of the two summands is strictly positive. In the above formula $f \in C^\infty(M)$ is such that $\psi^*\alpha = f\alpha$, and confer Subsection 2.3 for the expression of the Hamiltonian. Let us show the existence of the contactomorphism ψ.

Let $\varepsilon \in \mathbb{R}^+$ be small enough, $(\theta, x, y) \in \mathbb{S}^1 \times D^2_\varepsilon$ local coordinates and $g : \mathbb{S}^1 \times D^2_\varepsilon \longrightarrow \mathrm{Op}(\gamma)$ a local chart such that $\ker g^*\alpha_{st} = \ker\{d\theta + x dy\}$. It suffices to construct the compactly supported contactomorphism ψ in this local model $\mathbb{S}^1 \times D^2_\varepsilon$. The contact vector field ∂_y is generated by the Hamiltonian $H(\theta; x, y) = x$. This Hamiltonian can be cut–off to a smooth Hamiltonian

$$\widetilde{H} : \mathbb{S}^1 \times D^2_\varepsilon \longrightarrow \mathbb{R} \text{ such that } \widetilde{H} = H \text{ on } \mathbb{S}^1 \times D^2_{\varepsilon/4}$$

$$\text{and } \widetilde{H} = 0 \text{ on } \mathbb{S}^1 \times (D^2_\varepsilon \setminus D^2_{3\varepsilon/4}).$$

The flow generated by $\widetilde{H}$ exists for $\tau \in \mathbb{R}^+$ small enough, and for one such τ we can define the contactomorphism ψ to be the τ–time flow. $\square$

PROPOSITION 4.2. *The loop* $\delta_t = \zeta_t \circ (\beta_t \odot \overset{k)}{\cdots} \odot \beta_t)$ *in* $\mathrm{Cont}_0(\mathbb{S}^1 \times \mathbb{S}^2, \xi_{st})$ *is locally autonomous at any point of the open set* $\mathring{\mathbb{T}}_1$ *and positive on* $(\mathbb{S}^1 \times \mathbb{S}^2, \xi_{st})$ *for* $k \in \mathbb{Z}^+$ *large enough.*

PROOF. The loop ζ_t preserves the decomposition $\mathbb{S}^1 \times \mathbb{S}^2 = \mathbb{T}_1 \cup \mathbb{T}_2$. The Hamiltonian associated to the loop $\{\beta_t \odot \overset{k)}{\cdots} \odot \beta_t\}$ is the smooth function $kH(\beta_{kt})$. The Hamiltonian $H(\beta_t)$ is positive on $\mathbb{T}^2$ and thus for k large enough

$$H(\delta_t)(p,t) = z(p) + kH(\beta_{kt})(\zeta_t^{-1}(p), t) \geq -1 + kH(\beta_{kt})(\zeta_t^{-1}(p), t) > 0.$$

Therefore the Hamiltonian $H(\delta_t)$ is positive in $\mathbb{T}_2$.

In the solid torus $\mathbb{T}_1$, the Hamiltonian $H(\delta_t)|_{\mathbb{T}_1}(\theta; r, \varphi, z) = z + 2kr^2$ is positive, autonomous and its flow preserves $\mathbb{T}_1$. This concludes the statement. $\square$

The existence of the loop $\delta_t \in \Omega\,\mathrm{Cont}(\mathbb{S}^1 \times \mathbb{S}^2, \xi_{st})$ in Proposition 4.2 implies Theorem 1.1.

Proof of Theorem 1.1: The loop of contactomorphisms δ_t constructed in Proposition 4.2 is positive and locally autonomous at $p = (0; 0, 0, 1)$. Consider the transverse knot $\gamma = \{z = 1\} = \{(\theta; 0, 0, 1)\}$, this is coincides with the orbit of δ_t at p. Then the loop of contactomorphisms $\phi_t \# \delta_t$ of the fibered connected sum $(M, \xi) \# (\mathbb{S}^1 \times \mathbb{S}^2, \xi_{st})$ along $\kappa \# \gamma$ is generated by a positive Hamiltonian. Subsection 2.2 implies that the construction does not depend on the choice of frames and the resulting contact manifold is (M, ξ^κ). $\square$

The geometric argument used to prove Theorem 1.1 should apply to higher dimensional contact manifolds. There is however no explicit example of an overtwisted contact manifold of higher dimension and thus there is no local model in order to glue (neither a general notion of a higher dimensional Lutz twist).

CHAPTER 6

Non-existence of small positive loops on overtwisted contact 3–folds

In this sixth chapter we prove that in an overtwisted contact manifold there can be no positive loops of contactomorphisms that are generated by a $\mathcal{C}^0$–small Hamiltonian function. This work is developed in collaboration with F. Presas and M. Sandon.

1. Introduction

In 2000 Eliashberg and Polterovich [**55**] noticed that the natural notion of positive contact isotopies, i.e. contact isotopies that move every point in a direction positively transverse to the contact distribution, induces for certain contact manifolds a partial order on the universal cover of the contactomorphism group. Such contact manifolds are called *orderable*. Since the work of Eliashberg and Polterovich orderability has become an important subject in the study of contact topology. In particular it has been discovered to be deeply related to the contact non–squeezing phenomenon [**53**] (see also the exposition [**72**]) and, more recently, to the non–degeneracy of a natural bi–invariant metric that is defined on the universal cover of the contactomorphism group [**39**].

As Eliashberg and Polterovich explained, orderability of a contact manifold is equivalent to the non–existence of a positive contractible loop of contactomorphisms. By now many contact manifolds are known to be orderable and many are known not to be, but it is still not well–understood where the boundary between the orderable and non–orderable world lies. In particular it is not known whether there is a relation between overtwistedness and orderability, since not a single overtwisted contact manifold is known to be orderable or not to be. In this Chapter we prove the following result.

THEOREM 1.1. *Let $\big(M, \xi = \ker\alpha\big)$ be a closed overtwisted contact 3–manifold. Then there exists a real positive constant $C(\alpha)$ such that any positive loop $\{\phi_\theta\}$ of contactomorphisms which is generated by a contact Hamiltonian $H : M \times \mathbb{S}^1 \longrightarrow \mathbb{R}^+$ satisfies*

$$\|H\|_{\mathcal{C}^0} \geq C(\alpha)\,.$$

In other words, on closed overtwisted contact 3–manifolds there are no positive loops of contactomorphisms that are generated by a $\mathcal{C}^0$–small contact Hamiltonian. Note that there is no loss of generality in assuming that the contact Hamiltonian is 1–periodic, see Lemma 3.1.A in [**55**]. It is important to notice that our result does not imply that overtwisted contact manifolds are orderable, because the contraction of a positive contractible loop of contactomorphisms is not necessarily performed via positive loops. For instance, it was even proved in [**53**, Theorem 1.11] that for the standard tight contact sphere any contraction of a positive contractible loop must be sufficiently negative somewhere. Theorem 1.1 states though that there exists a lower bound for a Hamiltonian function that generates a positive loop of contactomorphisms. Intuitively, in the presence of an overtwisted disc a positive isotopy returning to the identity requires a minimal amount of energy.

The specificity of our result is that we deal with $\mathcal{C}^0$*–small* contact Hamiltonians. Indeed, let us prove that the non–existence of a positive loop of contactomorphisms that is generated by a $\mathcal{C}^1$*–small* Hamiltonian holds on any contact manifold. Consider first the $\mathcal{C}^2$–small case. If the Hamiltonian $H_\theta : M \longrightarrow \mathbb{R}$ is $\mathcal{C}^2$–small then the generated loop $\{\phi_\theta\}$ is $\mathcal{C}^1$–small and so the contact graphs[1] $\mathrm{gr}(\phi_\theta)$ are Legendrian sections in a Weinstein neighborhood of the diagonal Δ in the contact product $M \times M \times \mathbb{R}$. Since a Weinstein neighborhood is contactomorphic to a neighborhood of the zero section of $J^1(\Delta) = J^1(M)$, the graphs $\mathrm{gr}(\phi_\theta)$ are of the form $\{j^1 f_\theta\}$ for a family of smooth functions f_θ on M. Because of the Hamilton–Jacobi equation (see [**4**, Section 46]), positivity of the loop $\{\phi_\theta\}$ implies that the family f_θ is strictly increasing, yielding a contradiction. If the Hamiltonian function is only $\mathcal{C}^1$–small, and thus the loop $\{\phi_\theta\}$ is $\mathcal{C}^0$–small, then the graphs $\mathrm{gr}(\phi_\theta)$ are still contained in a Weinstein neighborhood of the diagonal in the contact product but they are not necessarily sections anymore, and so they cannot be written as 1-jet of functions. However it follows from Chekanov theorem [**33, 32**] that they have generating functions quadratic at infinity and so an argument similar to the one above (or the results in [**38, 34**]) allows to conclude also in this case. As far as we know, Theorem 1.1 is the first result in the literature that shows the non–existence of a positive loop in the case when the Hamiltonian is $\mathcal{C}^0$–small. Our proof strongly uses overtwistedness in several points, and does not give an intuition of

[1] See for example [**115, 39**] for the definition of contact graphs, contact products and more details on arguments similar to the one that follows.

whether or not the result should also be true for tight contact manifolds. However it seems plausible to us that this might be the case.

Although Theorem 1.1 only applies to overtwisted contact 3–manifolds, a higher–dimensional analogue can also be stated. The careful reader can try to generalize the result to non–fillable contact manifolds containing a PS–structure [**106, 113**], a GPS–structure [**107**] or a blob [**97**] with the appropriate hypotheses on the Chern class of the contact distributions. The precise statement is not part of this Chapter due to its technicality and to the fact that no new geometric ideas are required for the argument to work. We would also like to remark that the constant $C(\alpha)$ appearing in the statement of Theorem 1.1 can be explicitly computed with an algorithm. This procedure is however quite involved and requires an explicit symbolic solution for a system of differential equations as well as the expression of the flow of a vector field. In particular, a numerical estimate for $C(\alpha)$ is hardly attainable and hence we do not discuss this point further in the Chapter.

As a consequence of Theorem 1.1, we can bound from below not only the supremum norm of the Hamiltonian of a positive loop but also its L^1–norm, in the following sense.

Corollary 1.1. *Let $\big(M, \xi = \ker\alpha\big)$ be a closed overtwisted contact 3–manifold. Then there exists a real positive constant $C(\alpha)$ such that any positive loop of contactomorphisms $\{\phi_\theta\}$ which is generated by a contact Hamiltonian H_θ, $\theta \in \mathbb{S}^1$, satisfies*

$$\int_0^1 \|H_\theta\|_{\mathcal{C}^0}\, d\theta \geq C(\alpha)\,.$$

Corollary 1.1 can be deduced from Theorem 1.1 as follows. Suppose that there is a positive loop $\{\phi_\theta\}$ which is generated by a contact Hamiltonian H_θ that satisfies

$$\int_0^1 \|H_\theta\|_{\mathcal{C}^0}\, d\theta \leq C(\alpha)\,.$$

Define a reparametrization $\beta : [0,1] \longrightarrow [0,1]$ of the time–coordinate by requiring $\dot\beta(\theta) = \frac{\|H_\theta\|_{\mathcal{C}^0}}{\int_0^1 \|H_\theta\|_{\mathcal{C}^0}\, d\theta}$ and write $\psi_{\beta(\theta)} = \phi_\theta$. Then we have $H_\theta = \dot\beta(\theta) G_{\beta(\theta)}$, where $G_{\beta(\theta)}$ is the Hamiltonian of the reparametrized loop $\psi_{\beta(\theta)}$. For all $\theta \in \mathbb{S}^1$ we have

$$\max_x G_{\beta(\theta)}(x) = \frac{\int_0^1 \|H_\theta\|_{\mathcal{C}^0}\, d\theta}{\|H_\theta\|_{\mathcal{C}^0}} \|H_\theta\|_{\mathcal{C}^0} = \int_0^1 \|H_\theta\|_{\mathcal{C}^0}\, d\theta \leq C(\alpha)$$

contradicting Theorem 1.1.

The geometric core of the proof of Theorem 1.1 can be shortly described in two parts. First, any overtwisted contact manifold (M, ξ) can be embedded with trivial symplectic normal bundle in an exact symplectically fillable contact 5–manifold (X, ξ_X). Second, the existence of a small positive loop of contactomorphisms on (M, ξ) implies the existence of a PS–structure on (X, ξ_X). This yields a contradiction, according to the main result of [**106**]. The construction of a PS–structure on X is based on techniques similar to those used by Niederkrüger and the second author [**107**] to study the size of tubular neighborhoods of contact submanifolds.

The paper is organized as follows. Section 2 recalls basic definitions and facts about overtwisted contact manifolds. In Section 3 we explain how to construct a PS–structure in the total space of the contact fibration $M \times \mathbb{D}^2$, where M is an overtwisted contact 3–manifold, starting from a small positive loop of contactomorphisms of M. Theorem 1.1 is proved in Section 4 assuming an embedding result that will be proved in Section 5.

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2. Preliminaries on overtwisted contact manifolds

We refer to the book of H. Geiges [**61**] for the basics about contact structures, and recall here only the definitions and facts about overtwisted contact manifolds that will be needed in the rest of the Chapter. A 3–dimensional contact manifold (M, ξ) is said to be *overtwisted* if it contains an overtwisted disc, i.e. an embedded 2–disc Δ such that the characteristic foliation $T\Delta \cap \xi$ contains a unique singular point in the interior of Δ and $\partial\Delta$ is the only closed leaf of this foliation. A contact manifold is said to be tight if it is not overtwisted.

As follows from the results of Lutz and Martinet [**91, 94**], there exists an overtwisted contact structure in any homotopy class of 2–plane fields. Moreover, by the classification of overtwisted contact structures achieved by Eliashberg [**45**], we also know that on a given homotopy class of 2–plane fields there exists exactly one overtwisted contact structure. More precisely we have the following result.

THEOREM 2.1 ([**45**]). *Let ξ and ξ' be overtwisted contact structures on a 3–dimensional manifold M, and suppose that they are homotopic as 2–plane fields. Then ξ and ξ' are isotopic contact structures.*

The notion of an overtwisted contact structure does not readily generalize to higher–dimensional contact manifolds. The following geometric model was proposed by Niederkrüger [**106**].

DEFINITION 2.1. Let (M, ξ) be a contact 5–manifold. A *plastikstufe* $\mathrm{PS}(\mathbb{S}^1)$ in M with singular set $\mathbb{S}^1$ is an embedding of a solid torus

$$\iota : \mathbb{D}^2 \times \mathbb{S}^1 \longrightarrow M$$

with the following properties:

a. The boundary $\partial\mathbb{D}^2 \times \mathbb{S}^1$ is the unique closed leaf of the foliation $\ker(\iota^*\alpha)$ on $\mathbb{D}^2 \times \mathbb{S}^1$.
b. The interior of $\mathbb{D}^2 \times \mathbb{S}^1$ is foliated by an $\mathbb{S}^1$–family of stripes $(0,1) \times \mathbb{S}^1$ spanned between $\mathbb{S}^1 \times \{0\}$ and asymptotically approaching $\partial\mathbb{D}^2 \times \mathbb{S}^1$ on the other side.

In particular, Property a. implies that the boundary of the solid torus is a Legendrian torus and the core $\{0\} \times \mathbb{S}^1$ is transverse to the contact distribution ξ.

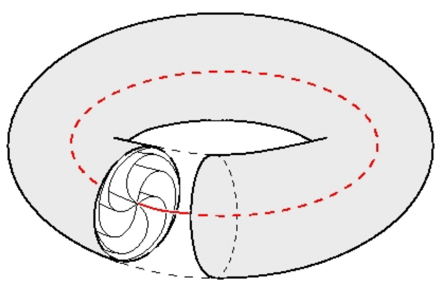

FIGURE 1. An embedded PS–structure in a contact 5–fold.

A plastikstufe is also referred to in the literature as a PS–structure.

By results of Gromov and Eliashberg [**73, 46**], in dimension 3 the presence of an overtwisted disc obstructs the existence of symplectic fillings. The higher–dimensional analogue of this fact is the following theorem by Niederkrüger.

THEOREM 2.2 ([**106**]). *Let (M, ξ) be a contact 5–manifold with a PS–structure. Then M does not admit an exact symplectic filling.*

As we will explain, the argument used to prove Theorem 1.1 is based on the insertion of a PS–structure in an exact symplectically fillable manifold, thus yielding a contradiction with Theorem 2.2. The techniques that provide the embedding of the PS–structure are based on the study of certain contact structures on the manifold $M \times \mathbb{D}^2$. This will be explained in the next section.

3. PS–structures and contact fibrations

In the first part of this section we will recall, following Lerman [**88**] and [**113**], the notion of a contact fibration and its relation to the group of contactomorphisms via the monodromy diffeomorphism. We will then show in Proposition 3.4 how to apply these concepts in order to construct a PS–structure on the total space of the contact fibration $M \times \mathbb{D}^2$, starting from a sufficiently small positive loop of contactomorphisms of M.

A smooth fiber bundle $\pi : X \longrightarrow B$ is said to be a *contact fibration* if there exists a hyperplane distribution $\xi_X = \ker \alpha_X$ on X such that its restriction $\xi = \ker(\pi) \cap \xi_X$ defines a contact structure in each fiber. In particular $\big(\xi, d\alpha_X|_{\ker(\pi)}\big)$ is a subbundle of the not necessarily symplectic bundle ξ_X. Note that ξ is itself a symplectic bundle. This data leads to a natural choice of connection.

DEFINITION 3.1. Let $\pi : \big(X, \xi_X = \ker \alpha_X\big) \longrightarrow B$ be a contact fibration. Then the distribution $\xi^{\perp d\alpha_X} \subset \xi_X$ is called the *contact connection* associated to the contact fibration.

In other words, for a point p of B and a tangent vector $v \in T_pB$, the horizontal lift of v at some $\tilde{p} \in \pi^{-1}(p)$ with respect to the contact connection is the unique vector $\tilde{v} \in T_{\tilde{p}}X$ such that $\pi_*\tilde{v} = v$, $\tilde{v} \in \ker(\xi_X)$ and $\iota_{\tilde{v}} d\alpha_X = 0$ on ξ. Note that the contact connection only depends on ξ_X, not on the choice of the 1–form α_X with $\xi_X = \ker \alpha_X$. The parallel transport along a segment joining two points $q, p \in B$ is defined as in the smooth case, but in the contact framework it is enhanced from a diffeomorphism to a contactomorphism between the fibers of q and p. Moreover, the definition of the contact connection implies that the trace by parallel transport of a submanifold that is tangent to the contact structure on the fibers is also tangent to the distribution on the total space. A precise statement of these properties is the content of the following proposition.

PROPOSITION 3.2. ([**88, 113**]) *Let $\pi : \big(X, \xi_X = \ker \alpha_X\big) \longrightarrow B$ be a contact fibration with closed fibers. Consider a point $p \in B$ and an immersed path $\gamma : [0,1] \longrightarrow B$ with $\gamma(0) = p$. Then parallel transport along γ with respect to the contact connection defines a path of diffeomorphisms*

$$\widetilde{\gamma}_t : \pi^{-1}(p) \longrightarrow \pi^{-1}(\gamma(t))$$

with the following properties:

a. *The diffeomorphisms* $\widetilde{\gamma}_t$ *are contactomorphisms.*
b. *Let* L *be an isotropic submanifold of* $\pi^{-1}(p)$ *and consider the map*

$$\mathfrak{t} : L \times [0,1] \longrightarrow X, \quad (p,t) \longmapsto \widetilde{\gamma}_t(p),$$

then $im(\mathfrak{t})$ *is an immersed isotropic submanifold of* (X, ξ_X)*. It is an embedded isotropic submanifold if* γ *is an embedded path.*

Note that the closedness condition for the fibers is technical and only used to ensure that the vector fields implicitly appearing in the statement are complete.

There are instances in which the contactomorphisms generated via parallel transport have a simple description. The following example will be used in the proof of our results.

Let $\big(M, \xi = \ker\alpha\big)$ be a contact manifold. A time–dependent function H_θ on M induces a path of contactomorphisms $\{\phi_\theta\}$, which is defined to be the flow of the time–dependent vector field X_θ satisfying

$$\begin{aligned} \iota_{X_\theta}\alpha &= H_\theta, \\ \iota_{X_\theta}d\alpha &= -dH_\theta + dH_\theta(R_\alpha)\,\alpha \end{aligned} \tag{3.1}$$

where R_α is the Reeb vector field associated to α. The function H_θ is called the *contact Hamiltonian* with respect to the contact form α of the contact isotopy $\{\phi_\theta\}$. In contrast to the symplectic case, any contact isotopy can be written as the flow of a contact Hamiltonian, see [**61**, Section 2.3].

Consider the manifold $M \times \mathbb{D}^2$, where $\mathbb{D}^2$ denotes the 2–disc with polar coordinates (r,θ). Let $H : M \times \mathbb{D}^2 \longrightarrow \mathbb{R}$ be a function such that $H \in O(r^2)$ at the origin and $\partial_r H > 0$. Then the 1–form

$$\alpha_H = \alpha + H(p,r,\theta)d\theta$$

defines a contact structure ξ_H on the manifold $M \times \mathbb{D}^2$. In particular, suppose that $H : M \times \mathbb{S}^1 \longrightarrow \mathbb{R}$ is a positive function. Then $\alpha_H = \alpha + H(p,\theta)\cdot r^2 d\theta$ is a contact form in $M \times \mathbb{D}^2$.

LEMMA 3.3. *Let* $\big(M, \xi = \ker\alpha\big)$ *be a contact manifold, and* $H_\theta : M \longrightarrow \mathbb{R}$ *an* $\mathbb{S}^1$*–family of positive smooth functions. Consider the contact fibration*

$$\pi : \big(M \times \mathbb{D}^2, \ker\alpha_H\big) \longrightarrow \mathbb{D}^2.$$

Then parallel transport along $\gamma(\theta) = (1, -\theta)$ *is the contact flow of the Hamiltonian* H_θ*.*

PROOF. The horizontal lift with respect to the contact connection of the vector field ∂_θ at a point $(1, \theta)$ is of the form

$$\widetilde{X} = \partial_\theta - X_\theta,$$

where X_θ satisfies the equations $\iota_{X_\theta}\alpha = H_\theta$ and $\iota_{X_\theta} d\alpha = -dH_\theta + dH_\theta(R_\alpha)\,\alpha$. Indeed, the lift is unique and $\widetilde{X}$ satisfies both $\alpha_H(\widetilde{X}) = 0$ and $\iota_{\widetilde{X}} d\alpha_H = 0$ on ξ. The statement then follows from equations (3.1). $\square$

Let us explain how to use Lemma 3.3 to construct a PS–structure in $\big(M \times \mathbb{D}^2(\delta), \ker(\alpha + r^2 d\theta)\big)$, where $\mathbb{D}^2(\delta)$ denotes the 2–disc of radius δ, using a positive loop of contactomorphisms in M.

PROPOSITION 3.4. *Assume that $\{\phi_\theta\}$ is a positive loop of contactomorphisms of an overtwisted contact manifold $\big(M, \xi = \ker\alpha\big)$ which is generated by a contact Hamiltonian H_θ, $\theta \in \mathbb{S}^1$, with $H_\theta < \delta^2$ for some $\delta \in \mathbb{R}^+$. Then there is a PS–structure on $\big(M \times \mathbb{D}^2(\delta), \ker(\alpha + r^2 d\theta)\big)$.*

PROOF. Note first the following general fact. Suppose that that $\pi : (X, \xi_X) \longrightarrow \Sigma$ is a contact fibration over a smooth compact surface Σ, such that the fibers are closed overtwisted contact manifolds. Suppose also that there exists an embedded loop $\gamma : \mathbb{S}^1 \longrightarrow \Sigma$ whose time–1 parallel transport $\widetilde{\gamma}_1$ is the identity. Then there exists a PS–structure in the pre–image $\pi^{-1}(\gamma(\mathbb{S}^1))$. Indeed, since the fiber $\pi^{-1}(\gamma(0))$ is overtwisted we can consider an embedded overtwisted disc Δ in it and define the map

$$\begin{aligned} \rho : \Delta \times \mathbb{S}^1 &\longrightarrow X \\ (p, \theta) &\longmapsto \rho(r, \theta) = \widetilde{\gamma}_\theta(p). \end{aligned}$$

Then property b. in Proposition 3.2 implies that $im(\rho)$ is a PS–structure. By combining this fact with Lemma 3.3 we see that if $\{\phi_\theta\}$ is a positive loop of contactomorphisms on a contact manifold $\big(M, \xi = \ker\alpha\big)$ then there is a PS–structure on $\big(M \times \mathbb{D}^2(1), \ker(\alpha + H_\theta r^2 d\theta)\big)$ where H_θ is the Hamiltonian function of $\{\phi_\theta\}$. The PS–structure lies in the boundary defined by the equation $\{r = 1\}$. Note that if $H_\theta < \delta^2$ for some $\delta \in \mathbb{R}^+$ then there exists a strict contact embedding

$$\big(M \times \mathbb{D}^2(1), \ker(\alpha + H_\theta r^2 d\theta)\big) \longrightarrow \big(M \times \mathbb{D}^2(\delta), \ker(\alpha + r^2 d\theta)\big)$$

given by the map $(p, r, \theta) \longmapsto (p, \sqrt{H_\theta(p)}r, \theta)$. A PS–structure in $\big(M \times \mathbb{D}^2(1), \ker(\alpha + H_\theta r^2 d\theta)\big)$ contained in the hypersurface defined as $\{r = 1\}$ is sent to a PS–structure in $\big(M \times \mathbb{D}^2(\delta), \ker(\alpha + r^2 d\theta)\big)$ contained in the hypersurface defined as $\{r = \sqrt{H_\theta}\}$. We have thus obtained the required PS–structure in $\big(M \times \mathbb{D}^2(\delta), \ker(\alpha + r^2 d\theta)\big)$. $\square$

4. Proof of Theorem 1.1

In this section we prove Theorem 1.1 in the case when $c_1(\xi) = 0$. As we will explain, the general case also follows from the same argument modulo Proposition 4.2 that will be proved in the last section.

Let (M, ξ) be a 3–dimensional overtwisted contact manifold and assume that $\{\phi_\theta\}$ is a positive loop of contactomorphisms, generated by a contact Hamiltonian H_θ, $\theta \in \mathbb{S}^1$. We want to show that if H_θ is small in $\mathcal{C}^0$–norm then the existence of $\{\phi_\theta\}$ gives a contradiction with Theorem 2.2.

Recall from the previous section that if $\{\phi_t\}$ is a positive loop of contactomorphisms of M which is generated by a sufficiently small contact Hamiltonian H_θ ($\theta \in \mathbb{S}^1$) then there is a PS–structure on $\big(M \times \mathbb{D}^2(\delta), \ker(\alpha + r^2 d\theta)\big)$ for some small $\delta \in \mathbb{R}^+$. Note that the manifold $\big(M \times \mathbb{D}^2(\delta), \ker(\alpha + r^2 d\theta)\big)$ is the standard contact neighborhood of a codimension–2 contact submanifold with trivial symplectic normal bundle, see [**61**, Section 2.5.3]. The result of the previous section implies thus the following proposition.

PROPOSITION 4.1. *Let (X, ξ_X) be a contact 5–manifold and (M, ξ) a codimension–2 overtwisted contact 3–manifold with trivial symplectic normal bundle. Suppose that $\{\phi_\theta\}$ is a positive loop of contactomorphisms which is generated by a sufficiently small contact Hamiltonian. Then there exists a PS–structure in a neighborhood of M in X.*

In the case when $c_1(\xi) = 0$, Theorem 1.1 follows from Proposition 4.1. Indeed any contact manifold (M, ξ) can be embedded as a contact submanifold into its unit cotangent bundle $\mathbb{S}T^*M$, by using the map $e_\alpha : M \longrightarrow \mathbb{S}T^*M$ defined by $e_\alpha(p) = \big(p, \alpha_p\big)$ where α is a contact form for ξ. Note that if $c_1(\xi) = 0$ then $e_\alpha(M) \subset \mathbb{S}T^*M$ has trivial symplectic normal bundle. Certainly, the symplectic normal bundle of $e_\alpha(M)$ inside $\mathbb{S}T^*M$ is isomorphic to ξ, and thus it is trivial if its Euler class $c_1(\xi)$ vanishes. If there was a small positive contact Hamiltonian H_θ that generated a loop of contactomorphisms then Proposition 4.1 would give a PS–structure inside $\mathbb{S}T^*M$. But the existence of a PS–structure inside $\mathbb{S}T^*M$ is impossible by Theorem 2.2 because $\mathbb{S}T^*M$ is an exact symplectically fillable manifold, a filling being given by $\mathbb{D}T^*M$. More precisely, a tubular neighborhood of $e_\alpha(M)$ inside $\mathbb{S}T^*M$ is contactomorphic to $\big(M \times \mathbb{D}^2(\delta), \ker(\alpha + r^2 d\theta)\big)$ for some $\delta > 0$. If the $\mathcal{C}^0$–norm of H_θ is smaller than δ^2 then we would obtain a PS–structure inside $\mathbb{S}T^*M$. The square of the maximal size δ of a tubular neighborhood $M \times \mathbb{D}^2(\delta)$ of M inside $\mathbb{S}T^*M$ gives in this case the constant $C(\alpha)$ that appears in the statement of Theorem 1.1.

In the general case, i.e. when $c_1(\xi)$ does not necessarily vanish, the proof of Theorem 1.1 follows from the same argument, combined with the following proposition.

PROPOSITION 4.2. *Every 3–dimensional overtwisted contact manifold can be embedded as a contact submanifold with trivial symplectic normal bundle into an exact symplectically fillable contact 5–manifold.*

The proof of this result will be given in the next section. Assuming it, Theorem 1.1 is proved as follows. Given an overtwisted contact 3–manifold (M, ξ), by Proposition 4.2 it can be embedded with trivial symplectic normal bundle into an exact symplectically fillable contact 5–manifold (X, ξ_X). If there was a sufficiently small contact Hamiltonian H_θ generating a positive loop of contactomorphisms then Proposition 4.1 would give a PS–structure in X, contradicting Theorem 2.2.

5. Contact embeddings with trivial normal bundle

In this section we will prove Proposition 4.2, i.e. that every overtwisted contact 3–manifold (M, ξ) can be embedded with trivial symplectic normal bundle into an exact symplectically fillable contact 5–manifold (X, ξ_X). The idea of the proof is to start with a contact embedding of (M, ξ) into its unit cotangent bundle $\mathbb{S}T^*M$ and then perform contact surgeries in an appropriate way in order to make the symplectic normal bundle trivial while keeping the symplectic fillability of the resulting 5–manifold. As we will see the process will also modify the contact structure on the initial overtwisted 3–manifold M. One of the crucial points of the proof will be to make sure that the modified contact structure on the 3–manifold will still be overtwisted and moreover in the same homotopy class as cooriented 2–plane fields as the initial one. Then it will be isotopic to it according to Eliashberg's classification theorem, confer Theorem 2.1.

We start by briefly recalling the notion of a Lutz twist, its effect on the homotopy class of the contact structure and its relation to contact surgery and symplectic cobordism. See [**61**] for more details on these notions.

Let K be a positive transverse knot in (M, ξ). A *Lutz twist*[2] along K is an operation that changes (see [**61**, Section 4.3]) the contact structure in a neighborhood of K. The resulting contact structure ξ^K on M is always overtwisted. The effect of a Lutz twist on the homotopy class of the contact structure can be described as follows. Given two 2–plane fields ξ_0 and ξ_1 there are two cohomology classes $d^2(\xi_0, \xi_1) \in H^2(M, \mathbb{Z})$ and $d^3(\xi_0, \xi_1) \in H^3(M, \mathbb{Z})$ that measure the obstruction for ξ_0 and ξ_1 to belong to the same homotopy class of plane fields. We refer to [**61**, Section 4.3] for details and for a proof of the following results.

PROPOSITION 5.1. *Let $K \subset M$ be a positive transverse knot on ξ. Then $d^2(\xi, \xi^K) = -pd([K])$.*

PROPOSITION 5.2. *Let $K \subset M$ be a null–homologous positive transverse knot on ξ with self–linking number $sl(K)$. Then $d^2(\xi, \xi^K) = 0$ and $d^3(\xi, \xi^K) = sl(K)$.*

Following Eliashberg [**47**] and Weinstein [**123**], fix a Legendrian knot L on a contact manifold 3–manifold M and fix the relative (-1)–framing with respect to the canonical contact framing associated to the knot. If we perform on M a handle attachment along L, then the resulting cobordism has a natural symplectic structure. The bottom boundary of this cobordism, i.e. the initial contact manifold, is a concave boundary of the symplectic structure. The upper boundary is convex and therefore it has an induced contact structure, which is said to be obtained from the initial one by contact (-1)–surgery. The inverse operation is called a contact $(+1)$–surgery.

As proved by Ding, Geiges and Stipsicz [**41**], the effect of a Lutz twist on a contact manifold can be described in terms of contact surgery as follows. Given a Legendrian knot $L \subset (M, \xi)$, denote by $t(L)$ a positive transverse push–off of L and by $\sigma(L)$ a Legendrian push–off of L with two added zig–zags. Then we have the following result.

PROPOSITION 5.3 ([**41**]). *Let (M, ξ) be a contact 3–manifold and L a Legendrian knot for ξ. The contact structure obtained by a Lutz twist along $t(L)$ is isotopic to the contact structure resulting from a contact $(+1)$–surgery along L and $\sigma(L)$.*

Being the inverse of a (-1)–surgery, the contact $(+1)$–surgery in a contact 3–fold corresponds to a symplectic 2–handle attachment to the concave boundary of a bounded part of the symplectization, i.e. we obtain

[2]We consider a Lutz twist what is called a half Lutz twist in other places of the literature.

a symplectic cobordism in which the new boundary is concave. Consider the transverse knot $K = t(L) \subset (M, \xi^K)$ and the belt spheres $\lambda_K, \lambda_K^\sigma \subset (M, \xi^K)$ corresponding to the contact $(+1)$–surgeries along L and $\sigma(L)$ in (M, ξ) described in Proposition 5.3. Then λ_K and λ_K^σ are two Legendrian knots in (M, ξ^K). Since Proposition 5.3 is a local result, both Legendrian knots can be assumed arbitrarily close to K. The following observation will be used in our argument.

Lemma 5.4. $[K] = [L] = [\lambda_K]$.

Proof. By definition $[K] = [t(L)] = [L]$. The equality $[L] = [\lambda_K]$ follows from the fact that the surgery in Proposition 5.3 is smoothly trivial. This implies the statement. See Proposition 6.4.5 in [**61**] for further details. □

A consequence of the description in Proposition 5.3 is the existence of an exact symplectic cobordism realizing a Lutz twist. More precisely we have the following result.

Corollary 5.5. *Let (M, ξ) be a contact 3–manifold and $K = t(L)$ a positive transverse knot which is a positive transverse push–off of a Legendrian knot L. Then there exists an exact symplectic cobordism (W, ω) from (M, ξ^K) to (M, ξ), which is realized by a 2–handle attachment along the Legendrian link $\lambda_K \cup \lambda_K^\sigma$.*

The convex end of (W, ω) is the contact boundary (M, ξ), the concave end is (M, ξ^K). A Lutz untwist is thus tantamount to an exact symplectic cobordism. It is central to note that the convex end of an exact symplectic cobordism is exact symplectically fillable if the concave end is. This fact will be crucial in our proof of Proposition 4.2, because it will ensure that the 5–manifold X into which we will embed (M, ξ) will still be fillable. Indeed, as we will see, X will be obtained by constructing an exact symplectic cobordism between contact 5–manifolds with an exact symplectically fillable concave end. This cobordism will restrict to a cobordism between contact 3–manifolds as the one described in Corollary 5.5.

In our argument we will also use the following result.

Lemma 5.6. *Let (M, ξ) be an overtwisted contact 3–manifold and Δ a fixed overtwisted disc. Consider a Legendrian link L in M disjoint from Δ. Then there exists a Legendrian link Λ disjoint from $L \cup \Delta$ such that ξ is isotopic to $\xi^{t(L \cup \Lambda)}$.*

Proof. Consider a Legendrian link $\widetilde{L}$ disjoint from $L \cup \Delta$ and with homology class $[\widetilde{L}] = -[L]$. Then Proposition 5.1 implies that $d^2(\xi, \xi^{t(L\cup\widetilde{L})}) = 0$. Let K be a null–homologous knot contained in a Darboux ball with self–linking number $-d^3(\xi, \xi^{t(L\cup\widetilde{L})})$. Propositions 5.1 and 5.2 imply that $\Lambda = \widetilde{L} \cup K$ satisfies $d^2(\xi, \xi^{t(L\cup\Lambda)}) = 0$ and $d^3(\xi, \xi^{t(L\cup\Lambda)}) = 0$. Theorem 2.1 concludes the statement of the Lemma. □

We are now almost ready to state and prove two results, Propositions 5.8 and 5.9, that will be the two main steps in the proof of Proposition 4.2. Proposition 5.8 will be an adaptation to higher dimensions of Proposition 5.3. We first discuss the smooth model for it.

Denote by $M_L(\tau)$ the manifold obtained by surgery along a knot $L \subset M$ with framing τ. In case this is a contact surgery along a Legendrian knot, the notation stands for a contact (-1)–surgery. The following observation is a strictly differential topological statement.

Lemma 5.7. *Let X be a smooth 5–manifold and M a codimension–2 submanifold. Consider a knot L in M and a framing τ of L in X. Suppose that τ restricts to a framing τ_s of L in M. Then a surgery on X along L with framing τ induces a surgery on M along L with framing τ_s.*

Proof. The statement can be seen as a consequence of the description of a surgery as a handle attachment. The gradient flow used to glue a 6–dimensional 2–handle $H^6 \cong \mathbb{D}^2 \times \mathbb{D}^4$ along the attaching sphere L in $X \times \{1\} \subset X \times [0,1]$ restricts to a gradient flow in the submanifold $M \times \{1\}$. This describes the attachment of a 4–dimensional 2–handle $H^4 \cong \mathbb{D}^2 \times \mathbb{D}^2$ along L in $M \times \{1\} \subset M \times [0,1]$. Note that the belt 3–sphere in the handle H^6 intersects the surgered submanifold $M_L(\tau_s)$ along the belt 1–sphere of the handle H^4. □

Lemma 5.7 provides the smooth model for the symplectic cobordism we shall construct to prove Proposition 4.2. Proposition 5.3 concerns contact 3–manifolds and a 4–dimensional symplectic cobordism. In view of Lemma 5.7 we can adapt Proposition 5.3 to the context of a codimension–2 contact submanifold in a contact 5–manifold. The result is as follows.

Proposition 5.8. *Let (X, ξ_X) be a contact 5–manifold and (M, ξ) a codimension–2 overtwisted contact submanifold. Consider a transverse knot $K = t(L)$ in M which is the core of a Lutz tube and such that $c_1(\nu_M) = pd([\lambda_K])$. Denote $\lambda = \lambda_K \cup \lambda_K^\sigma$. Then there exists a framing*

τ of λ in (X, ξ_X) restricting to the Legendrian framing τ_s of λ in (M, ξ) such that $M_\lambda(\tau_s)$ is contactomorphic to M with a Lutz untwist along K, and the symplectic normal bundle of $M_\lambda(\tau_s)$ in $X_\lambda(\tau)$ is trivial.

Proof. The contact (-1)–surgery that occurs on the contact 3–manifold (M, ξ) is the procedure described in Proposition 5.3 and Corollary 5.5. It suffices to explain the choice of framing τ for the link λ in X. The Legendrian framing τ_s for λ in (M, ξ) is extended to a framing τ for λ in X. This extension is obtained as follows.

Consider a section $\mathfrak{s} : M \longrightarrow \nu_M$ transverse to the 0–section and such that

$$\lambda_K = Z(\mathfrak{s}), \text{ where } Z(\mathfrak{s}) = \{p \in M : \mathfrak{s}(p) = 0\}.$$

This section exists since $c_1(\nu_M) = pd([\lambda_K])$. It is used to define the extension of the Legendrian framing τ_s to τ for λ_K. Let us discuss this in detail and the effect of the surgery. It can be considered in two stages. First, surgery along the Legendrian link λ_K. The required framing along λ_K is defined to be $\tau = (\tau_s, \mathfrak{s}_* \tau_s)$. Thus τ is constructed using the differential $\mathfrak{s}_*$ of the section $\mathfrak{s}$. The section $\mathfrak{s}$ cannot be used since it vanishes along λ_K. Consider polar coordinates $(r, w_1, w_2) \in \mathbb{D}^4 \subset \mathbb{C}^2$ with $(w_1, w_2) \in \mathbb{S}^3$. The framing τ provides a diffeomorphism

$$f_\tau : \mathbb{S}^1 \times \mathbb{D}^2 \times \mathbb{D}^2 \longrightarrow \mathcal{U}(\lambda_K) \subset X, \quad (\theta; r, w_1, w_2) \longmapsto f_\tau(\theta; r, w_1, w_2)$$

and we can suppose that $f_\tau(\mathbb{S}^1 \times \mathbb{D}^2 \times \{0\}) = \mathcal{U}(\lambda_K) \cap M$, for a neighborhood $\mathcal{U}(\lambda_K)$ of $\lambda_K \subset X$. The differential $\mathfrak{s}_*$ identifies the pull–back $f_\tau^*(\nu_M)$ of the normal bundle with the trivial bundle $\mathbb{C} \longrightarrow \mathbb{S}^1 \times \mathbb{D}^2 \times \{0\}$ over a neighborhood of $\lambda_K \subset M$. We can also suppose that the section $\mathfrak{s}$ in these local coordinates is $((f_\tau)^*\mathfrak{s})(\theta; r, w_1) = rw_1$. The function rw_1 is well–defined although the coordinate w_1 is not well–defined at $r = 0$.

The surgery substitutes the core $\lambda_K \cong \mathbb{S}^1 \times \{0\} \times \{0\} \subset \mathbb{S}^1 \times \mathbb{D}^2 \times \mathbb{D}^2$ with coordinates $(\theta; r, w_1, w_2)$ by $\{0\} \times \mathbb{S}^3 \subset \mathbb{D}^2 \times \mathbb{S}^3$ with coordinates $(r, \theta; w_1, w_2)$ along the common boundary $\mathbb{S}^1 \times \mathbb{S}^3 = \{(r, \theta; w_1, w_2) : r = 1\}$. The section $((f_\tau)^*\mathfrak{s})(\theta; r, w_1) = rw_1$ can be substituted by a section of the form

$$g : \mathbb{D}^2 \times \mathbb{S}^3 \longrightarrow \mathbb{C}, \quad (r, \theta; w_1, w_2) \longmapsto g(r, \theta; w_1, w_2) = \rho(r) w_1$$

where $\rho : \mathbb{R} \longrightarrow \mathbb{R}^+$ is a positive smooth function. In particular it is non–vanishing and provides a trivialization of the normal bundle of the surgered submanifold $M_{\lambda_K}(\tau_s)$ in the surgered manifold $X_{\lambda_K}(\tau)$.

Second, surgery along the Legendrian link λ_K^σ. The manifold $M_{\lambda_K}(\tau_s)$ has trivial normal bundle in $X_{\lambda_K}(\tau)$. Thus there exists a global framing

τ_ν of this normal bundle. Denote the restriction of this global framing τ_ν to the Legendrian knot λ_K^σ by $\tau_\nu|_{\lambda_K^\sigma}$. Then the framing $\{\tau_s, \tau_\nu|_{\lambda_K^\sigma}\}$ is a framing of the normal bundle of λ_K^σ inside $X_{\lambda_K}(\tau)$. Thus, once the surgery along λ_K^σ is performed with the framing $\{\tau_s, \tau|_{\lambda_K^\sigma}\}$, the resulting normal bundle is still trivial. Denote by τ the extended framing as described above for both λ_K and λ_K^σ. Then the normal bundle of $M_\lambda(\tau_s)$ in $X_\lambda(\tau)$ is trivial. $\square$

A minor modification of the argument for Proposition 5.8 yields the following result.

PROPOSITION 5.9. *Let (X, ξ_X) be a contact 5–manifold and (M, ξ) a codimension–2 overtwisted contact submanifold with trivial normal bundle. Consider a transverse knot $K = t(L)$ in M which is the core of a Lutz tube and denote $\lambda = \lambda_K \cup \lambda_K^\sigma$. Then there exists a framing τ of λ in (X, ξ_X) restricting to the Legendrian framing τ_s of λ in (M, ξ) such that $M_\lambda(\tau_s)$ is contactomorphic to M with a Lutz untwist along K and the symplectic normal bundle of $M_\lambda(\tau_s)$ in $X_\lambda(\tau)$ is trivial.*

PROOF. In this case there is no need to use $\mathfrak{s}_*$ since the section $\mathfrak{s}$ can be chosen to be non–vanishing. Thus we choose the framing described in the second part of the surgery in Proposition 5.8. Id est, the framing induced by $\mathfrak{s}$. The surgery along λ with this framing preserves the triviality of the normal bundle. $\square$

We are now ready to prove Proposition 4.2.

Let (M, ξ) be an overtwisted contact 3–manifold. We want to show that there is a contact embedding with trivial symplectic normal bundle of (M, ξ) into an exact symplectically fillable contact 5–manifold.

Fix an overtwisted disc Δ in (M, ξ) and take a Legendrian link L in (M, ξ) which is disjoint from Δ and such that $pd([L]) = c_1(\xi)$. By Lemma 5.6 we know that there exists a Legendrian link Λ in (M, ξ) disjoint from L and Δ and such that ξ is isotopic to $\overline{\xi} := \xi^{t(L\cup\Lambda)}$. Consider a contact embedding $(M, \overline{\xi}) \longrightarrow \mathbb{S}T^*M$ defined by some contact form $\overline{\alpha}$ for $\overline{\xi}$. The symplectic normal bundle of this embedding is isomorphic to $\overline{\xi}$ and hence to ξ. Note that L and Λ are still Legendrian in $(M, \overline{\xi})$. Consider the transverse push–offs, with respect to $\overline{\xi}$, $K = t(L)$ and $\kappa = t(\Lambda)$.

First, we apply Proposition 5.8 to $(M, \overline{\xi})$ inside $(X, \xi_X) := \mathbb{S}T^*M$, and $K = t(L)$. We can apply it because the symplectic normal bundle of

$(M, \overline{\xi})$ inside (X, ξ_X) is $\overline{\xi}$ and we know that

$$c_1(\overline{\xi}) = c_1(\xi) = pd([L]) = pd([K]) = pd([\lambda_K]).$$

The last equality holds by Lemma 5.4. After applying Proposition 5.8 we get contact structures $\overline{\xi}'$ on M and ξ'_X on X such that $(M, \overline{\xi}')$ embeds into (X, ξ'_X) with trivial symplectic normal bundle, and $\overline{\xi}'$, ξ'_X are obtained from $\overline{\xi}$, ξ_X by performing a Lutz untwist along K.

Second, consider $\kappa = t(\Lambda)$ as a transverse link in $(M, \overline{\xi}')$ and apply Proposition 5.9 to $(M, \overline{\xi}')$ inside (X, ξ'_X) and $\kappa = t(\Lambda)$. We obtain contact structures $\overline{\xi}''$ on M and ξ''_X on X such that $(M, \overline{\xi}'')$ embeds into (X, ξ''_X) with trivial symplectic normal bundle and $\overline{\xi}''$, ξ''_X are obtained from $\overline{\xi}'$, ξ'_X by performing a Lutz untwist along κ.

Recall that $\overline{\xi}$ was obtained from ξ by performing a Lutz twist along $K \cup \kappa = t(L \cup \Lambda)$. We have thus that $\overline{\xi}''$ and ξ are in the same homotopy class. Since the overtwisted disc has not been affected by the previous operations, Theorem 2.1 implies that the two contact structures $\overline{\xi}''$ and ξ are actually isomorphic. We have thus obtained an embedding

$$(M, \xi) \longrightarrow (X, \xi''_X)$$

with trivial symplectic normal bundle. The contact manifold (X, ξ_X) is exact symplectically fillable, it follows from Corollary 5.5 and the discussion after it that (X, ξ''_X) is also exact symplectically fillable. This finishes the proof of Proposition 4.2 and hence the proof of Theorem 1.1 in the general case.

CHAPTER 7

A Remark on the Reeb Flow for Spheres

In this seventh chapter we prove the non–triviality of the Reeb flow for the standard contact spheres $\mathbb{S}^{2n+1}$, $n \neq 3$, inside the fundamental group of their contactomorphism group. The argument uses the existence of homotopically non–trivial 2–spheres in the space of contact structures of a 3–Sasakian manifold. The results in this chapter are joint work with F. Presas.

1. Introduction

Let (M, ξ) be a closed contact manifold. Consider the space $\mathcal{C}(M, \xi)$ of contact structures isotopic to ξ. This space has been studied in special cases. See [**49**] for the 3–sphere and [**20**], [**65**] for torus bundles. In the present note we prove the non–triviality of its second homotopy group for 3–Sasakian manifolds, see [**23**].

THEOREM 1.1. *Let (M, ξ) be a 3–Sasakian manifold, then the rank satisfies* $\mathrm{rk}(\pi_2(\mathcal{C}(M, \xi))) \geq 1$.

Let $(\mathbb{S}^{4n+3}, \xi_0 = \ker \alpha_0)$ be the standard contact sphere with the standard contact form. The non–trivial spheres in $\mathcal{C}(\mathbb{S}^{4n+3}, \xi_0)$ allow us to answer a question posed in [**72**]:

Remarque 2.10: On peut se demander s'il n'y a pas, dans $\mathrm{Cont}(\mathbb{S}^{2n+1}, \xi_0)$*, un lacet positif contractile plus simple que dans* $\mathbb{P}U(n, 1)$ *et par exemple si le lacet ρ_t, $t \in \mathbb{S}^1$, n'est pas contractile. C'est peu probable mais je n'en ai pas la preuve.*

The answer we provide is the following

COROLLARY 1.1. *The class in* $\pi_1(\mathrm{Cont}(\mathbb{S}^{2n+1}, \xi_0))$ *generated by the Reeb flow of α_0 is a non–trivial element of infinite order for $n \neq 3$.*

In Section 2 we introduce the objects of interest and necessary notation. The geometric construction underlying the results is explained in Section 3. It is a generalization to higher dimensions of ideals found in [**65**].

Theorem 1.1 is concluded. Section 4 contains the argument deducing Corollary 1.1. Section 5 extends the results to higher homotopy groups.

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2. Preliminaries

2.1. Contact structures.

Definition 2.1. Let M^{2n+1} be a smooth manifold. A codimension–1 regular distribution ξ is a contact distribution if there exists a 1–form $\alpha \in \Omega^1(M)$ such that $\ker \alpha = \xi$ and $\alpha \wedge d\alpha^n$ is a volume form.

The structure described above is known as a cooriented contact structure. Since the non–coorientable case is not considered in this Chapter, we refer to a cooriented contact structure simply as a contact structure. The smooth manifold M will be assumed to be oriented. The contact structures to be considered will be positively cooriented, i.e. the induced orientation coincides with that prescribed on M.

The definition is independent of the choice of 1–form $\alpha' = e^f\alpha$, for any $f \in C^\infty(M,\mathbb{R})$. Let $\mathrm{Cont}(M,\xi) = \{s \in \mathrm{Diff}(M) : ds_*\xi = \xi\}$ be the space of diffeomorphisms that preserve the contact structure. These diffeomorphisms are called contactomorphisms. The connected component of the identity of $\mathrm{Cont}(M,\xi)$ will be denoted by $\mathrm{Cont}_0(M,\xi)$. $\mathcal{C}(M,\xi)$ will stand for the space of positive contact structures in M isotopic to ξ. The unique vector field R such that

$$i_R\alpha = 1, \quad i_R d\alpha = 0,$$

is called the Reeb vector field associated to α.

A vector field $X \in \Gamma(TM)$ preserves the contact structure if it satisfies the following pair of equations

$$\begin{aligned} i_X\alpha &= H, \\ i_X d\alpha &= -dH + (i_R dH)\alpha, \end{aligned}$$

for a choice of α and a function $H \in C^\infty(M,\mathbb{R})$. Such a function is called the Hamiltonian associated to the vector field. This correspondance defines a linear isomorphism between the space of vector fields $\Gamma_\xi(TM)$

preserving the contact structure ξ and the vector space of smooth functions $C^\infty(M,\mathbb{R})$. By definition, a contactomorphism $\phi \in \mathrm{Cont}_0(M,\xi)$ admits an expression as $\phi = \phi_1$ for a time dependent flow $\{\phi_t\}_{t\in[0,1]}$ generated by a time dependent family $X_t \in \Gamma_\xi(TM)$. Therefore, its flow $\{\phi_t\}$ can be generated by a time dependent family of smooth functions $\{H_t\}$.

2.2. Contact fibrations. A smooth fibration $\pi : X \longrightarrow B$ is said to be contact for a codimension–1 distribution $\xi \subset TX$ if for any fiber $F_p = \pi^{-1}(p) \overset{e}{\hookrightarrow} X$, the restriction of the distribution $e^*\xi$ is a contact structure on the fibre. We assume that the distribution ξ is cooriented. Any $\alpha \in \Omega^1(X)$ such that $\xi = \ker\alpha$ will be referred to as a fibration form.

Let $\pi : X \longrightarrow B$ be a smooth fibration. The vertical subbundle $V \subset TX$ is defined fiberwise by $V_x = \ker d\pi(x), \forall x \in X$. An *Ehresmann connection* is a smooth choice of a fiberwise complementary linear space H_x for V_x inside T_xX. Therefore, the map $d\pi_x : H_x \longrightarrow TB_{\pi(x)}$ is a linear isomorphism and there is a well-defined notion of parallel transport.

There is a canonical connection once a contact fibration $(\pi, \xi = \ker\alpha)$ is fixed. The connection H is defined at a point $x \in X$ to be the annihilator of the vector subspace $V_x \cap \xi_x$ with respect to the quadratic form $(\xi, d\alpha)$. It is complementary to V_x since $V_x \cap \xi_x$ is a symplectic space for the 2–form $d\alpha$. The connection is independent of the choice of fibration form α. See [**113**] for details on the following facts.

LEMMA 2.2. *The parallel transport of the canonical connection associated to a contact fibration is by contactomorphisms.*

LEMMA 2.3. *Let $(F, \ker\alpha_0)$ be a closed contact manifold. Let $\pi : F \times \mathbb{D}^2 \longrightarrow \mathbb{D}^2$ be a contact fibration with fibration distribution defined by the kernel of $\alpha = \alpha_0 + Hd\theta$, for some function $H : F \times \mathbb{D}^2 \longrightarrow \mathbb{R}$ satisfying $|H| = O(r^2)$. Fix a loop $\gamma : \mathbb{S}^1 \longrightarrow \mathbb{D}^2$, defined as $\gamma(\theta) = \gamma(r_0, \theta)$ in polar coordinates. Then, the contactomorphism of the fiber $F \times (r_0, 0)$ defined by the parallel transport along γ is generated by the family of Hamiltonian functions $\{G_\theta(p) = -H(p, r_0, \theta)\}_{\theta\in[0,2\pi]}$.*

Let us study general contact fibrations over a 2–disk $\mathbb{D}^2$. Fix a contact fibration $\pi : X \longrightarrow \mathbb{D}^2$ with distribution $\xi = \ker\alpha$. Consider the radial vector field $Y = \partial_r$, defined on $\mathbb{D}^2 \setminus \{0\}$. It can be lifted to X by using the canonical contact connection. This produces a vector field $\widetilde{Y} : X \setminus F_0 \longrightarrow TX$. Once an angle θ_0 is fixed it can be uniquely extended to $0 \in \mathbb{D}^2$. In such a case, denote by $\phi_{r,\theta_0} : F_0 \longrightarrow F_{(r,\theta_0)}$ the associated flow at time r. It identifies via contactomorphisms the fibers over $0 \in \mathbb{D}^2$ and over $(r, \theta_0) \in \mathbb{D}^2$. Define the diffeomorphism:

$$\begin{array}{rcl} \Phi : F_0 \times D^2 & \longrightarrow & X \\ (p, r, \theta) & \longmapsto & \phi_{r,\theta}(p). \end{array}$$

Then the definition of the contact connection implies $\Phi^*\alpha = e^g(\alpha_0 + Hd\theta)$, where $g : M \times D^2 \longrightarrow \mathbb{R}$ and $H : M \times D^2 \longrightarrow \mathbb{R}$ are arbitrary smooth functions. We can choose as fibration form $\alpha' = e^{-g}\alpha$ and trivialize the fibration using Φ. Then we obtain the expression

$$\Phi^*\alpha' = (\alpha_0 + Hd\theta). \tag{2.1}$$

Given a contact fibration over the disk, the trivialization constructed above is called *radial*. It is convenient to observe that the radial trivialization construction can be made parametric for families of contact fibrations over the disk.

2.3. Loops at infinity. Fix a contact fibration $\pi : X \longrightarrow \mathbb{S}^2$ with distribution ξ, fibre F and a point $N \in \mathbb{S}^2$. This point will be referred to as North pole or infinity. Define the restriction fibration $\pi_N : X \setminus \pi^{-1}(N) \longrightarrow \mathbb{S}^2 \setminus N \simeq \mathbb{D}^2$. Trivialize the contact fibration π_N *radially* from $S = \{0\} \in \mathbb{D}^2$ to obtain a new contact fibration $\hat{\pi} : F \times \mathbb{D}^2 \longrightarrow \mathbb{D}^2$ with fibration form $\alpha_0 + Hd\theta$. Denoting by $\Phi : F \times \mathbb{D}^2 \longrightarrow X \setminus \pi^{-1}(N)$ the trivialization map, we obtain $\Phi^*\xi = \ker\{\alpha_0 + Hd\theta\}$. Therefore, the map is connection–preserving. Consider the family of loops

$$\begin{array}{rcl} \gamma_r : \mathbb{S}^1 & \longrightarrow & D^2 \\ \theta & \longmapsto & (r, \theta). \end{array}$$

Composing with the embedding $\mathbb{D}^2 \hookrightarrow \mathbb{S}^2$, for $r \longrightarrow 1$, they are smaller and smaller loops around the North pole $N \in \mathbb{S}^2$. By Lemma 2.3, the parallel transport associated to the loop γ_r is generated by a family of Hamiltonians $\{G^r_\theta\}_{\theta\in\mathbb{S}^1}$, defined by $G^r_\theta(p) = -H(p, r, \theta)$. The limit function

$$G_\theta = \lim_{r\longrightarrow 1} G^r_\theta$$

exists because the connection associated to ξ is a smooth connection over $\mathbb{S}^2$. It is clear that $\{G_\theta\}$ defines a loop in $\mathrm{Cont}(M, \xi_0 = \ker\alpha_0)$. This will be called the *loop at infinity* associated to (π, ξ). Continuous families of contact fibrations with marked fibre produce continuous families of loops at infinity.

DEFINITION 2.4. A contact sphere is a smooth map $e : \mathbb{S}^2 \longrightarrow \mathcal{C}(M, \xi)$.

There is a canonical contact fibration over $\mathbb{S}^2$ associated to any contact sphere e. It is defined as

$$X = M \times \mathbb{S}^2 \longrightarrow \mathbb{S}^2,$$

with the distribution at $(p, z) \in M \times \mathbb{S}^2$ being $\xi^e(p, z) = e(z)_p \oplus T_z\mathbb{S}^2 \subset T_pM \oplus T_z\mathbb{S}^2$.

Denote by $C^\infty(\mathbb{S}^2, \mathcal{C}(M, \xi))$ the space of smooth maps from $\mathbb{S}^2$ to $\mathcal{C}(M, \xi)$. The smooth loop space of $\mathrm{Cont}_0(M, \xi)$ is denoted as $\Omega(\mathrm{Cont}_0(M, \xi), id)$.

LEMMA 2.5. *The previous construction induces a continuous map*

$$C^\infty(\mathbb{S}^2, \mathcal{C}(M, \xi)) \longrightarrow \Omega(\mathrm{Cont}_0(M, \xi), id).$$

Therefore, it provides a morphism

$$\pi_2(\mathcal{C}(M, \xi)) \longrightarrow \pi_1(\mathrm{Cont}_0(M, \xi)).$$

2.4. Homotopy sequence. The group $\mathrm{Diff}_0(M)$ acts transitively on $\mathcal{C}(M, \xi)$ because of Gray's Stability Theorem. It is a Serre fibration with homotopy fibre $\mathrm{Cont}(M, \xi) \cap \mathrm{Diff}_0(M)$. This homotopy fibre might be disconnected. Its identity component is denoted by $\mathrm{Cont}_0(M, \xi)$. Hence the fibration induces a long exact sequence

$$\ldots \longrightarrow \pi_2(\mathrm{Diff}_0(M)) \longrightarrow \pi_2(\mathcal{C}(M, \xi)) \xrightarrow{\partial_2} \tag{2.2}$$

$$\xrightarrow{\partial_2} \pi_1(\mathrm{Cont}_0(M, \xi)) \longrightarrow \pi_1(\mathrm{Diff}_0(M)) \longrightarrow \ldots .$$

The map ∂_2 is the one provided by Lemma 2.5. The study of this sequence will provide Corollary 1.1.

Note that a geometric lifting map

$$\pi_j(\mathcal{C}(M, \xi)) \xrightarrow{\partial_j} \pi_{j-1}(\mathrm{Cont}_0(M, \xi)) \tag{2.3}$$

can be analogously defined. It provides a geometric representative of the connecting morphism. This generalizes the previous constructions. It will be used in Section 5.

3. Spheres in $\mathcal{C}(M,\xi)$

3.1. Almost contact structures. Let M be an oriented $(2n+1)$–dimensional manifold. Denote by $Dist(M)$ the space of smooth codimension–1 regular cooriented distributions on M. Concerning orientations, an almost complex structure on a cooriented distribution will be positive if the induced orientation coincides with the prescribed one. Define the space of almost contact structures as

$$\mathcal{A}(M) = \{(\xi,\mathcal{J}) : \xi \in Dist(M), \mathcal{J} \in \mathrm{End}(\xi), \mathcal{J}^2 = -\mathrm{id}, \mathcal{J} \text{ positive}\}.$$

Given a contact structure $\xi = \ker\alpha$, an almost complex structure $\mathcal{J} \in \mathrm{End}(\xi)$ is said to be compatible with α if it is compatible with the symplectic form on the symplectic space $(\xi, d\alpha)$. The space $\mathcal{A}(M)$ has a subset defined by

$$\mathcal{AC}(M,\xi) = \{(\eta,\mathcal{J}) : \eta \in \mathcal{C}(M,\xi), \mathcal{J} \in \mathrm{End}(\eta),$$
$$\mathcal{J}^2 = -\mathrm{id}, \mathcal{J} \text{ compatible with } \alpha \text{ such that } \eta = \ker\alpha\}.$$

The space of almost complex structures compatible with a fixed symplectic form is contractible. Thus, the forgetful map $\mathcal{AC}(M,\xi) \longrightarrow \mathcal{C}(M,\xi)$ has a contractible homotopy fibre. Hence there exists a homotopy inverse $\imath : \mathcal{C}(M,\xi) \longrightarrow \mathcal{AC}(M,\xi)$ provided by the choice of a compatible almost complex structure on the contact distribution.

Fix a point $p \in M$ and an oriented framing $\tau : T_pM \xrightarrow{\simeq} \mathbb{R}^{2n+1}$. Define the evaluation map

$$e_{(p,\tau)} : \mathcal{A}(M) \longrightarrow \mathcal{A}(\mathbb{R}^{2n+1}), \quad e_{(p,\tau)}(\xi,\mathcal{J}) = (\tau_*\xi_p, \tau_*\mathcal{J}_p).$$

This is a continuous map and thus induces $\widetilde{e}_{(p,\tau)} : \pi_2(\mathcal{A}(M)) \longrightarrow \pi_2(\mathcal{A}(\mathbb{R}^{2n+1}))$. Therefore, we obtain

$$\varepsilon_{(p,\tau)} = \widetilde{e}_{(p,\tau)} \circ \imath_* : \pi_2(\mathcal{C}(M,\xi)) \longrightarrow \pi_2(\mathcal{A}(\mathbb{R}^{2n+1}))$$

Lemma 3.1. *$\pi_2(\mathcal{A}(\mathbb{R}^{2n+1})) \cong \mathbb{Z}$.*

Proof. The space $\mathcal{A}(\mathbb{R}^{2n+1})$ is isomorphic to the homogeneous space $SO(2n+1)/U(n)$. The standard inclusion $SO(2n) \longrightarrow SO(2n+1)$ descends to a map

$$SO(2n)/U(n) \longrightarrow SO(2n+1)/U(n)$$

with homotopy fibre $\mathbb{S}^{2n}$. The long exact sequence for a homotopy fibration implies that

$$\pi_2(SO(2n)/U(n)) \cong \pi_2(SO(2n+1)/U(n)), \quad n \geq 2.$$

It is simple to show that $SO(2n+1)/U(n)$ is also isomorphic to $SO(2n+2)/U(n+1)$. Since $SO(4)/U(2)$ is a 2–sphere, the statement follows. $\square$

Thus the evaluation map can be seen as an integer–valued map for $\pi_2(\mathcal{C}(M,\xi))$.

LEMMA 3.2. *The map* $\varepsilon_{(p,\tau)} : \pi_2(\mathcal{C}(M,\xi)) \longrightarrow \pi_2(\mathcal{A}(\mathbb{R}^{2n+1}))$ *is independent of the choice of* p *and* τ.

PROOF. Let $p, q \in M$ and τ_p,τ_q be oriented framings of T_pM, T_qM respectively. Consider a continuous path of pairs $\{(p_t,\tau_t)\}$ connecting (p,τ_p) and (q,τ_q). The continuous family of maps

$$e_{(p_t,\tau_t)} : \mathcal{A}(M) \longrightarrow \mathcal{A}(\mathbb{R}^{2n+1}), \quad e_{(p_t,\tau_t)}(\xi,\mathcal{J}) = (\tau_{t_*}\xi_p, \tau_{t_*}\mathcal{J}_p)$$

provides a homotopy between $e_{(p,\tau_p)}$ and $e_{(q,\tau_q)}$. $\square$

3.2. Linear Contact Spheres.

DEFINITION 3.3. A linear contact sphere is a contact sphere $\iota : \mathbb{S}^2 \longrightarrow \mathcal{C}(M,\xi)$ such that there exist three contact forms $(\alpha_0,\alpha_1,\alpha_2)$ satisfying

$$\iota(p) = \ker(e_0\alpha_0 + e_1\alpha_1 + e_2\alpha_2)$$

for the standard embedding $(e_0,e_1,e_2) : \mathbb{S}^2 \longrightarrow \mathbb{R}^3$.

REMARK 3.4. Such spheres can only exist in a $(4n+3)$–dimensional manifold. The fact that α and $-\alpha$ do not induce the same volume form in dimensions congruent to 1 modulo 4 yields an obstruction for their existence.

Note that for a 3–fold the triple $(\alpha_0,\alpha_1,\alpha_2)$ constitutes a framing of the cotangent bundle.

LEMMA 3.5. *Let* M *be a* 3*–fold and* S *a linear contact sphere. The class* $[S] \in \pi_2(\mathcal{C}(M,\xi))$ *is non–trivial and has infinite order.*

PROOF. Let $p \in M$ be a point and consider the framing $\tau = (\alpha_0,\alpha_1,\alpha_2)_p$. In the three–dimensional case $\mathcal{A}(\mathbb{R}^3)$ is homotopic to a 2–sphere. This homotopy can be realized by projection π onto the space of cooriented 2–plane distributions. The degree of the evaluation map is computed via

$$\mathbb{S}^2 \xrightarrow{\varepsilon_{(p,\tau)}} \mathcal{A}(T_pM) \xrightarrow{\pi} Dist(\mathbb{R}^3) \cong \mathbb{S}^2$$

$$z \longmapsto e_0(z)\alpha_0(p) + e_1(z)\alpha_1(p) + e_2(z)\alpha_2(p) \longmapsto (e_0(z), e_1(z), e_2(z)).$$

Being the identity, this map has degree 1. $\square$

3.3. **3–Sasakian manifolds.** Let us define a class of contact manifolds with natural linear contact spheres.

DEFINITION 3.6. Let (M^{4n+3}, g) be a Riemannian manifold. It is said to be 3–Sasakian if the holonomy group of the metric cone $(C(M), \bar{g}) = (M \times \mathbb{R}^+, r^2 g + dr \otimes dr)$ reduces to $Sp(n+1)$.

This implies that $(C(M), \bar{g})$ is a hyperkähler manifold $(C(M), \bar{g}, I, J, K)$. The hyperkähler structure induces a 2–sphere of complex structures

$$\mathbb{S}^2(\bar{g}) = \{e_0 I + e_1 J + e_2 K : e_0^2 + e_1^2 + e_2^2 = 1\}.$$

Any such complex structure $\mathcal{J} \in \mathbb{S}^2(\bar{g})$ endows $(M \times \mathbb{R}^+, \bar{g})$ with a Kähler structure, providing (M, g) with a Sasakian structure. The vertical vector field ∂_r on $M \times \mathbb{R}^+$ is orthogonal to $M \times \{1\}$ and the form α defined by $\alpha_{\mathcal{J}}(v) = g(v, \mathcal{J}\partial_r)$ is a contact structure. Thus, a 3–Sasakian structure provides a linear contact sphere $\{\alpha_{\mathcal{J}}\}_{\mathcal{J} \in \mathbb{S}^2(\bar{g})}$ generated by α_I, α_J and α_K.

THEOREM 3.1. *Let M^{4n+3} be a 3–Sasakian manifold. The class of the associated linear contact sphere is an element of infinite order in the second homotopy group $\pi_2(\mathcal{C}(M, \ker(\alpha_I)))$.*

PROOF. Let $p \in M$ and note that the $4n$–distribution $\eta = \ker \alpha_I \cap \ker \alpha_J \cap \alpha_K$ is (I, J, K)–invariant. Thus, it can be identified with the quaternionic vector space $\mathbb{H}^n$ by fixing a quaternionic framing $v = \{v_1, \ldots, v_n\}$. This induces a real framing $\tau = \{v, Iv, Jv, Kv\}$ for η, identifying it with $\mathbb{R}^{4n}$ endowed with the standard quaternionic structure.

Consider the Reeb vector fields R_I, R_J, R_K associated to α_I, α_J, and α_K. Extend the framing τ to $\widetilde{\tau} = \{\tau, R_I, R_J, R_K\}$. Interpret the space $\mathcal{A}(\mathbb{R}^{4n+3})$ as pairs of $(v, \mathcal{J})$, where $v \in \mathbb{S}^{4n+2} \subset \mathbb{R}^{4n+3}$ is a unit vector and $\mathcal{J}$ an almost complex structure in its orthogonal space. Define

$$h : \mathcal{A}(\mathbb{R}^{4n+3}) \longrightarrow \mathcal{J}(\mathbb{R}^{4n+3} \oplus \mathbb{R}), \tag{3.1}$$

$$(v, \mathcal{J}) \longmapsto \{\widetilde{\mathcal{J}} : \langle v \rangle^{\perp} \oplus \langle v \rangle \oplus \langle \partial_t \rangle \longrightarrow \langle v \rangle^{\perp} \oplus \langle v \rangle \oplus \langle \partial_t \rangle\}$$

where the almost complex structure is $\widetilde{\mathcal{J}} = \begin{pmatrix} \mathcal{J} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$. This induces a morphism of second homotopy groups. Through the above identification the linear contact sphere generated by $(\alpha_I, \alpha_J, \alpha_K)$ evaluates in a sphere $\langle (\xi_I, I), (\xi_J, J), (\xi_K, K) \rangle \in \mathcal{A}(\mathbb{R}^{4n+3})$. This sphere maps via

(3.1) to the sphere of complex structures generated by the triple (I, J, K) in $\mathcal{J}(\mathbb{R}^{4n+4})$.

It is left to prove that the class of that sphere is an infinite order element of $\pi_2(SO(4n+4)/U(2n+2))$. Let us write $m = n+1$ to ease the notation. The homotopy fibration

$$U(2m) \longrightarrow SO(4m) \longrightarrow SO(4m)/U(2m)$$

induces an injection $\pi_2(SO(4m)/U(2m)) \longrightarrow \pi_1(U(2m)) \cong \mathbb{Z}$.

Let $(\theta, \phi) \in [0, 2\pi] \times [0, \pi]$ be spherical angles. Define

$$J_\theta = \cos\theta J + \sin\theta K, \quad \widetilde{I} = \cos\phi I + \sin\phi J_\theta, \quad P_{\theta,\phi} = \cos(\phi/2) I + \sin(\phi/2) J_\theta.$$

The sphere is represented by $\widetilde{I}$, we shall compute its image under the boundary morphism. Note that $P_{\theta,\phi} \in SO(4m)$ and $\widetilde{I} = P^t_{\theta,\phi} I P_{\theta,\phi}$. Further $P_{\theta,\pi} = J_\theta = (cos\theta \cdot id + \sin\theta I)J$, with $cos\theta \cdot id + \sin\theta I \in U(2m)$ and $J \in SO(4m)$. This decomposition provides a representative in $\pi_2(SO(4m)/U(2m))$. Thus the loop in $\pi_1(U(2m))$ is provided by $cos\theta \cdot id + \sin\theta I$ with $\theta \in [0, 2\pi]$. Since the identification $\pi_1(U(2m)) \cong \mathbb{Z}$ is given by the complex determinant, the degree of the sphere is $2m$. □

The argument above applies to a broader class of manifolds:

Definition 3.7. A contact manifold (M, ξ_0) is said to possess an almost–quaternionic sphere if it admits a sphere $\mathbb{S}^2 \xrightarrow{\xi} \mathcal{C}(M, \xi_0)$ such that:

1) There exists a family $\{\mathcal{J}_p\}_{p\in\mathbb{S}^2}$ compatible with the contact distributions $\xi_p = \xi(p)$,
2) There exists a point $q \in M$ and a framing τ for T_qM such that $e_{q,\tau}(\xi(\mathbb{S}^2))$ becomes the linear sphere associated to

$$\langle(\xi_I, I), (\xi_J, J), (\xi_K, K)\rangle \in \mathcal{A}(\mathbb{R}^{4n+3}).$$

Corollary 3.8. *An almost–quaternionic sphere inside a contact manifold (M, ξ) generates a class of infinite order in $\pi_2(\mathcal{C}(M, \xi))$.*

4. Reeb Flow for Spheres

Let us prove Corollary 1.1. The standard contact sphere will be denoted $(\mathbb{S}^{2n+1}, \xi)$. The relevant case is that of the spheres $\mathbb{S}^{2k+1}$ with k odd. Indeed, for the spheres $\mathbb{S}^{2k+1}$ with $k = 2n$ the Reeb flow is non–trivial in $\pi_1(SO(4n+2)) \hookrightarrow \pi_1(\mathrm{Diff}_0(\mathbb{S}^{4n+1}))$. Thus it cannot be contractible in $\mathrm{Cont}_0(M, \xi) \subset \mathrm{Diff}_0(\mathbb{S}^{4n+1})$. In order to conclude the case $\mathbb{S}^{4n+3}$ we detail the construction in Sections 2 and 3.

Consider the endomorphisms I, J, K of $\mathbb{R}^{4(n+1)}$ obtained by direct sum of the corresponding endomorphisms i, j, k of $\mathbb{R}^4$, satisfying the quaternionic relations

$$i^2 = j^2 = k^2 = ijk = -1.$$

The endomorphisms I, J, K anti–commute and hence any of their linear combinations is a complex structure. Let $e = (e_0, e_1, e_2) : \mathbb{S}^2 \longrightarrow \mathbb{R}^3$ be the standard embedding of the 2–sphere in Euclidean 3–space with azimuthal angle θ and polar angle ϕ:

$$e_0 = \cos\theta \sin\phi, \quad e_1 = \sin\theta\sin\phi, \quad e_2 = \cos\phi, \quad (\theta, \phi) \in [0, 2\pi] \times [0, \pi].$$

A complex structure $\mathcal{J} \in End(\mathbb{R}^{4n+4})$ induces the real $(4n+2)$–distribution

$$\xi_{\mathcal{J}} = T\mathbb{S}^{4n+3} \cap \mathcal{J}T\mathbb{S}^{4n+3}$$

of $\mathcal{J}$–complex tangencies on the sphere $\mathbb{S}^{4n+3}$. There exists a unique, up to scaling, $U(\mathcal{J}, n)$–invariant 1–form $\alpha_{\mathcal{J}}$ such that $\ker\alpha_{\mathcal{J}} = \xi_{\mathcal{J}}$. It is given by $\alpha(z) = z^t\mathcal{J}dz$. We use the following three 1–forms

$$\alpha_0 = \alpha_I, \quad \alpha_1 = \alpha_J, \quad \alpha_2 = \alpha_K.$$

Their respective Reeb vector fields R_0, R_1 and R_2 are linearly independent and their flows are given by the family of rotations generated by I, J and K. Consider the 1–form $\alpha = e_0\alpha_0 + e_1\alpha_1 + e_2\alpha_2$. The form α is a contact form on $\mathbb{S}^{4n+3}$ for each value of e. Although not used in the rest of this Chapter, it is simple to prove the following

LEMMA 4.1. $(\mathbb{S}^2 \times \mathbb{S}^{4n+3}, \ker\alpha)$ *is a contact manifold.*

Let us compute the loop at infinity for the trivial contact fibration

$$\mathbb{S}^2 \times \mathbb{S}^{4n+3} \longrightarrow \mathbb{S}^2, (e, p) \longmapsto e.$$

In the spherical coordinates above, we will obtain the loop at infinity corresponding to $\phi = \pi$. The contact connection allows us to lift a vector field X in the base $\mathbb{S}^2$. The lift $\widetilde{X}$ is the unique vector field on $\mathbb{S}^2 \times \mathbb{S}^{4n+3}$ conforming the two conditions

$$\alpha(\widetilde{X}) = 0, \qquad d\alpha(\widetilde{X}, V) = 0, \quad \text{with } V \text{ an arbitrary vertical vector field.}$$

Since uniqueness is provided once a solution is found, the following assertion can be readily verified

LEMMA 4.2. *The lift of the polar vector field ∂_ϕ to the contact connection given by α is*

$$\widetilde{X}_\phi = \partial_\phi + \frac{1}{2}\left(-\sin\theta R_0 + \cos\theta R_1\right).$$

The Hamiltonian will appear once we pull–back the contact form α with the π–time flow of the lift $\widetilde{X}_\phi$. Consider the linear endomorphism $F_\theta = \frac{1}{2}(-\sin\theta I + \cos\theta J)$. The flow associated to $\widetilde{X}_\phi$ induces a diffeomorphism between the central fibre $\{\phi = 0\}$ and the fibre at an arbitrary ϕ. This diffeomorphism can be expressed as

$$\varphi_\phi : \mathbb{S}^{4n+3} \longrightarrow \mathbb{S}^{4n+3}, \quad \varphi(p) = e^{F_\theta \phi} p.$$

This is understood as a map in complex space $\mathbb{C}^{2n+2}$ restricted to the sphere. The theory explained in Section 2, in particular formula (2.1), implies that the pull–back will be of the form $\alpha_2 + H(p,\phi)d\theta$. A computation yields

LEMMA 4.3. $\varphi_\phi^*(\alpha) = \alpha_2 + \sin^2(\phi/2)d\theta$

The loops correspond to the flow of the vector field associated to $G = -\sin^2(\phi/2)$. The loop at infinity has Hamiltonian $G|_{\phi=\pi} \equiv -1$. Thus it is the Reeb flow.

We have geometrically realized the boundary map of the long exact homotopy sequence (2.2). The non–contractibility of the Reeb flow will follow from an understanding of the contact sphere above and the group $\pi_2(\mathrm{Diff}_0(\mathbb{S}^{4n+3}))$. Regarding the former we have the following

LEMMA 4.4. *Let S be the sphere of complex structures*

$$S = \{e_0 I + e_1 J + e_2 K : e \in \mathbb{S}^2\} \subset SO(4n+4)/U(2n+2).$$

1) *$[S]$ is a non–trivial element of $\pi_2(SO(4n+4)/U(2n+2)) \cong \mathbb{Z}$.*
2) *The image of S in $\mathcal{C}(\mathbb{S}^{4n+3}, \xi)$ generates an infinite cyclic subgroup in $\pi_2(\mathcal{C}(\mathbb{S}^{4n+3}, \xi))$.*

PROOF. Both statements follow from the argument provided in the proof of Theorem 3.1. □

Concerning the group $\mathrm{Diff}_0(\mathbb{S}^{4n+3})$, the following lemma will suffice.

LEMMA 4.5. $\pi_2(\mathrm{Diff}(\mathbb{S}^{4n+3})) \otimes \mathbb{Q} = 0$ *for $n \geq 2$.*

PROOF. This is a result in algebraic topology. Let $\mathrm{Diff}_0(\mathbb{D}^l, \partial)$ be the group of diffeomorphisms of the l–disk restricting to the identity at the boundary. Note the homotopy equivalence

$$\mathrm{Diff}_0(\mathbb{S}^l) \simeq SO(l+1) \times \mathrm{Diff}_0(\mathbb{D}^l, \partial)$$

and that $\pi_2(SO(l+1)) = 0$ since $SO(l+1)$ is a Lie group. Let $\phi(l) = \min\{(l-4)/3, (l-7)/2\}$. In the stable concordance range $0 \leq j < \phi(l)$ we have

$$\pi_j(\mathrm{Diff}_0(\mathbb{D}^l, \partial)) \otimes \mathbb{Q} = 0, \quad \text{if } l \text{ even or } 4 \not| j+1. \tag{4.1}$$

See [**123**] for details. In particular $\pi_2(\mathrm{Diff}_0(\mathbb{D}^l, \partial)) \otimes \mathbb{Q} = 0$ for $l > 11$. We are thus able to conclude

$$\pi_2(\mathrm{Diff}_0(\mathbb{S}^{4n+3})) \otimes \mathbb{Q} \cong \pi_2(\mathrm{Diff}_0(\mathbb{D}^{4n+3}, \partial)) \otimes \mathbb{Q} = 0, \quad n > 2.$$

For the case $n = 2$ we provide a more *ad hoc* argument. Let $C(\mathbb{D}^{11})$ be the space of pseudo–isotopies for the disk $\mathbb{D}^{11}$. There exists a homotopy fibration

$$\mathrm{Diff}_0(\mathbb{D}^{12}, \partial) \longrightarrow C(\mathbb{D}^{11}) \longrightarrow \mathrm{Diff}_0(\mathbb{D}^{11}, \partial)$$

Algebraic K–theory implies $\pi_1 C(\mathbb{D}^{11}) \otimes \mathbb{Q} = \pi_2 C(\mathbb{D}^{11}) \otimes \mathbb{Q} = 0$. Observe that (4.1) implies that $\pi_1(\mathrm{Diff}(\mathbb{D}^{12}, \partial))$ is a torsion group. The long exact homotopy sequence of the above fibration gives

$$\ldots \longrightarrow \pi_2(C(\mathbb{D}^{11})) \xrightarrow{\rho_2} \pi_2(\mathrm{Diff}_0(\mathbb{D}^{11}, \partial)) \xrightarrow{\partial}$$

$$\xrightarrow{\partial} \pi_1(\mathrm{Diff}_0(\mathbb{D}^{12}, \partial)) \xrightarrow{i_1} \pi_1(C(\mathbb{D}^{11})) \longrightarrow \ldots$$

This implies the short exact sequence of Abelian groups

$$0 \longrightarrow A \longrightarrow \pi_2(\mathrm{Diff}_0(\mathbb{D}^{11}, \partial)) \longrightarrow B \longrightarrow 0,$$

where $A = \ker \partial = \operatorname{im} \rho_2$ and $B = \operatorname{im} i_1 = \operatorname{coker} \rho_2$. Thus $\pi_2(\mathrm{Diff}_0(\mathbb{D}^{11}, \partial))$ is a torsion group. $\square$

REMARK 4.6. The Smale conjeture $\mathrm{Diff}_0(\mathbb{S}^3) \simeq SO(4)$ holds for $\mathbb{S}^3$, see [**78**].

In order to conclude Corollary 1.1 for $\mathbb{S}^{4n+3}$ consider the class of the Reeb loop in $\pi_1(\mathrm{Cont}_0(\mathbb{S}^{4n+3}, \xi))$. The construction explained above shows that it lies in the image of the boundary morphism

$$\partial_2 : \pi_2(\mathcal{C}(\mathbb{S}^{4n+3}, \xi)) \longrightarrow \pi_1(\mathrm{Cont}_0(\mathbb{S}^{4n+3}, \xi)).$$

If the Reeb class were to be zero the sphere S would lie in the image of $\pi_2(\mathrm{Diff}_0(\mathbb{S}^{4n+3}))$ in (2.2). Lemma 4.5 implies that such a sphere needs to be a torsion class if $n \geq 2$. Lemma 4.4 contradicts this statement.

Thus proving Corollary 1.1.

5. Higher homotopy groups

The previous arguments can be modified for n–dimensional homotopy spheres. This allows us to conclude properties of the higher homotopy type of the contactomorphism group. Consider the evalution map

$$e_{p,\tau} : \mathcal{A}(M) \longrightarrow \mathcal{A}(\mathbb{R}^{2n+1}).$$

Composition with the homotopy inverse $\imath : \mathcal{C}(M) \to \mathcal{AC}(M)$ defines higher homotopy maps

$$\pi_k(e_{p,\tau} \circ \imath) : \pi_k(\mathcal{C}(M)) \longrightarrow \pi_k(\mathcal{A}(\mathbb{R}^{2n+1})), \quad k \geq 1.$$

Let us provide a simple application. Define the natural inclusion

$$i_{\mathcal{J}} : \mathcal{J}(\mathbb{R}^{2n+2}) \longrightarrow \mathcal{C}(\mathbb{S}^{2n+1}, \xi), \quad \mathrm{i}_{\mathcal{J}}(\mathcal{J}) = T\mathbb{S}^{2n+1} \cap \mathcal{J}T\mathbb{S}^{2n+1}.$$

LEMMA 5.1. *The map $i_{\mathcal{J}}$ is a homotopy inclusion.*

PROOF. Consider the following chain of maps

$$c : \mathcal{J}(\mathbb{R}^{2n+2}) \xrightarrow{i_{\mathcal{J}}} \mathcal{C}(\mathbb{S}^{2n+1}, \xi) \xrightarrow{e_{p,\tau} \circ \imath} \mathcal{A}(\mathbb{R}^{2n+1}) \xrightarrow{h} \mathcal{J}(\mathbb{R}^{2n+2}).$$

The definition of each map implies $c = id$. Therefore, it induces the identity in homotopy:

$$\pi_k(c) = id : \pi_k(\mathcal{J}(\mathbb{R}^{2n+2})) \xrightarrow{\pi_k(i_{\mathcal{J}})} \pi_k(\mathcal{C}(\mathbb{S}^{2n+1}), \xi) \xrightarrow{\pi_k(e_{p,\tau} \circ \imath)}$$
$$\xrightarrow{\pi_k(e_{p,\tau} \circ \imath)} \pi_k(\mathcal{A}(\mathbb{R}^{2n+1})) \xrightarrow{\pi_k(h)} \pi_k(\mathcal{J}(\mathbb{R}^{2n+2})).$$

Thus the map $i_{\mathcal{J}}$ induces an injection $\pi_k(i_{\mathcal{J}})$, $\forall k \geq 0$. $\square$

This lemma can be combined with results on the homotopy type of the group $\mathrm{Diff}(\mathbb{S}^{2n+1})$. We can then conclude the existence of infinite order elements in certain homotopy groups of $\mathrm{Cont}(\mathbb{S}^{2n+1}, \xi)$. Among many others, a simple instance is the following

LEMMA 5.2. *The group $\pi_5(\mathrm{Cont}(\mathbb{S}^{2n-1}, \xi))$ has an element of infinite order, for $n \geq 11$.*

PROOF. Using the connecting map ∂_6, as described in equation (2.3), the statement is reduced to the following two assertions:

- $\pi_6(\mathcal{J}(\mathbb{R}^{2n})) = \pi_6(SO(2n)/U(n)) = \mathbb{Z}$ and therefore, by Lemma 5.1, $\mathrm{rk}(\pi_6(\mathcal{C}(\mathbb{S}^{2n-1}, \xi))) \geq 1$.

- $\pi_6(\mathrm{Diff}(\mathbb{S}^{2n-1})) \otimes \mathbb{Q} = 0$, for $n \geq 10$. This is again a consequence of the results in [**123**].

$\square$

Bott Periodicity Theorem allows us to apply the same argument to infinitely many other homotopy groups of $\mathrm{Cont}(\mathbb{S}^{2n-1}, \xi)$. These techniques can be adapted for general contact manifolds as long as there is a partial understanding of the homotopy type of their group of diffeomorphisms.

CHAPTER 8

Overtwisted Disks and Exotic Symplectic Structures

In this last chatper, the symplectization of an overtwisted contact $(\mathbb{R}^3, \xi_{ot})$ is shown to be an exotic symplectic $\mathbb{R}^4$. The technique can be extended to produce exotic symplectic $\mathbb{R}^{2n}$ using a GPS–structure and applies to symplectizations of appropriate open contact manifolds.

1. Introduction

Let $(\mathbb{R}^{2n}, \omega_0)$ be the standard symplectic structure on $\mathbb{R}^{2n}$. A symplectic structure ω on $\mathbb{R}^{2n}$ is exotic if there exists no symplectic embedding

$$\varphi : (\mathbb{R}^{2n}, \omega) \longrightarrow (\mathbb{R}^{2n}, \omega_0).$$

The non–existence of embedded exact Lagrangians in $(\mathbb{R}^{2n}, \omega_0)$ and the h–principle for immersions imply that $\mathbb{R}^{2n}$ admits an exotic symplectic structure for $n \geq 2$. See Exercise b. in page 344 in [**73**].

A symplectic structure on $\mathbb{R}^2$ is symplectomorphic to the standard symplectic structure. In the case of $\mathbb{R}^4$ and $\mathbb{R}^6$ exotic symplectic structures are provided in [**11**] and [**103**] respectively. The articles [**101, 119**] contain an approach to exotic Stein structures. Note that a finite type Stein manifold diffeomorphic to $\mathbb{R}^4$ has to be symplectomorphic to $(\mathbb{R}^4, \omega_0)$. The detection of exotic symplectic structures often relies on symplectic arguments, such as the study of embedded Lagrangians. See also [**124**].

The aim of the present Chapter is to show that techniques in contact topology can also be used to construct and detect exotic symplectic structures. In particular the exotic symplectic structures we describe are simple and explicit. The arguments we provide use known obstructions to fillability. See [**106, 107**]. The proofs in these articles require pseudo–holomorphic curves. This is the only place where non–elementary contact topology is invoked. The main result is the following

Theorem 1.1. *Let $(\mathbb{R}^3, \xi_{ot})$ be an overtwisted contact structure, then the symplectization $\mathcal{S}(\mathbb{R}^3, \xi_{ot})$ endows $\mathbb{R}^4$ with an exotic symplectic structure.*

EXAMPLE 1.1. Let $(\rho, \varphi, z) \in \mathbb{R}^3$ be cylindrical coordinates and $(\mathbb{R}^3, \xi_1)$ the contact structure defined by the kernel of the contact form

$$\alpha_1 = \cos \rho dz + \rho \sin \rho d\varphi.$$

Consider the symplectic 2–form $\omega_1 = d(e^t \alpha_1)$ on $\mathbb{R}^4 \cong \mathbb{R}^3 \times \mathbb{R}(t)$. Then $(\mathbb{R}^4, \omega_1)$ is an exotic symplectic structure. $\square$

The arguments we use apply to several open contact manifolds. For instance:

THEOREM 1.2. *Let (M, ξ) be an exact symplectically fillable contact 3–fold and $(U, \xi) \subset (M, \xi)$ an open contact submanifold. Consider an overtwisted contact structure (U, ξ_{ot}). Then $\mathcal{S}(U, \xi)$ is not symplectomorphic to $\mathcal{S}(U, \xi_{ot})$.*

The same techniques allow us to prove similar results in higher–dimensions. In particular we prove that the exotic symplectic structures obtained in Theorem 1.1 are stable.

THEOREM 1.3. *Let $(\mathbb{R}^3, \xi_{ot})$ be an overtwisted contact structure and $(\mathcal{S}(\mathbb{R}^3, \xi_{ot}), \omega_{ot})$ its symplectization. Then $(\mathcal{S}(\mathbb{R}^3, \xi_{ot}) \times \mathbb{R}^{2n-4}, \omega_{ot} + \omega_0)$ endows $\mathbb{R}^{2n}$ with an exotic symplectic structure.*

The appropriate analogue of Theorem 3 also holds for higher dimensions.

I have been informed that Y. Chekanov may have a different argument for Theorem 1.1. K. Niederkrüger explained to me that one can use bLobs as a generalization for the GPS–structure. The paper is organized as follows. Sections 2 and 3 introduce the ingredients used to prove the above results. The proof of Theorem 1.1 is detailed in Section 4. Section 5 contains the proof of Theorems 3 and 1.3 .

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2. Preliminaries

2.1. Contact structures on $\mathbb{R}^3$. The study of contact structures in $\mathbb{R}^3$ yielded to foundational work in contact topology. The first step towards an isomorphism classification was the distinction between the standard contact structure on $\mathbb{R}^3$ and the overtwisted contact structure described in Example 1.1. This is the work of D. Bennequin in [**13**]. The isomorphism classification of contact structures on $\mathbb{R}^3$ is completed after the seminal work of Y. Eliashberg in [**45, 49, 50**].

The standard contact structure ξ_0 on $\mathbb{R}^3(\rho, \varphi, z)$ is defined as the kernel of the contact form

$$\alpha_0 = dz + \rho^2 d\varphi.$$

This is a normal form of any contact 1–form in a sufficiently small neighborhood of a point in a contact 3–fold.

The contact structure ξ_1 induced by the contact form $\alpha_1 = \cos\rho dz + \rho \sin\rho d\varphi$ contains an overtwisted disk $\Delta = \{(\rho, \varphi, z) : \rho \leq \pi, z = 0\}$.The arguments in [**13**] imply that $(\mathbb{R}^3, \xi_0)$ and $(\mathbb{R}^3, \xi_1)$ are not contactomorphic.

Consider the 3–sphere $\mathbb{S}^3$. The main result in [**45**] implies the existence of a unique overtwisted contact structure in each homotopy class of plane distribution on $\mathbb{S}^3$. There are $H^3(\mathbb{S}^3, \pi_3(\mathbb{RP}^2)) = \mathbb{Z}$ homotopy classes. Denote by ζ_k the overtwisted contact structure in the homotopy class identified with $k \in \mathbb{Z}$. Then ζ_k restricted to $\mathbb{S}^3 \setminus \{p\}$, $p \in \mathbb{S}^3$, defines an overtwisted contact structure on $\mathbb{R}^3$. It will still be denoted ζ_k. The classification result in [**50**] is the following

THEOREM 2.1. *Each contact structure on $\mathbb{R}^3$ is isotopic to one of the structures ξ_0, ξ_1 or ζ_k, for $k \in \mathbb{Z}$. These structures are pairwise non–contactomorphic.*

Thus the overtwisted disk $\Delta \subset (\mathbb{R}^3, \xi_1)$ is the local model in a neighborhood of any overtwisted disk. That is, any small ball containing an overtwisted disk in a contact 3–fold is necessarily contactomorphic to $(\mathbb{R}^3, \alpha_1)$.

The symplectic structures we consider in this Chapter are constructed with a contact structure. The procedures we use to obtain a symplectic manifold from a contact manifold and viceversa will be explained in the following subsections. This material can be found in [**8**].

2.2. Symplectization. Let (M, ξ) be a contact manifold and $\mathcal{S}(M, \xi)$ be the subbundle of the cotangent bundle $\pi : T^*M \longrightarrow M$ whose fibre at a point $p \in M$ consists of all non–zero linear functions on the tangent space T_pM which vanish on the contact hyperplane $\xi_p \subset T_pM$ and define its given coorientation. Giving $\mathcal{S}(M,\xi)$ as a subbundle of the cotangent bundle T^*M is tantamount to endowing M with a contact structure.

Consider the Liouville 1–form λ on T^*M, the 2–form $d\lambda$ restricts to a symplectic structure on $\mathcal{S}(M,\xi)$.

DEFINITION 2.1. The symplectization of (M,ξ) is the exact symplectic manifold

$$(\mathcal{S}(M,\xi), d\lambda|_{\mathcal{S}(M,\xi)}).$$

In our perspective the primitive is not part of the data, only the symplectic structure is. In the study of Liouville domains the primitive is also part of the structure of a symplectization. This is not the case. The bundle $\pi : \mathcal{S}(M,\xi) \longrightarrow M$ is a trivial principal $\mathbb{R}^+$–bundle. The sections of π are contact forms for the contact structure ξ. A choice of contact form α defines a trivialization $\mathcal{S}(M,\xi) \cong M \times \mathbb{R}^+(t)$. In terms of this splitting $\lambda|_{\mathcal{S}(M,\xi)} = t\alpha$. In case a contact form α has been given to (M,ξ), its symplectization $\mathcal{S}(M,\xi)$ will also be denoted by $\mathcal{S}(M,\alpha)$. Contactomorphic contact manifolds yield symplectomorphic symplectizations.

In this Chapter $\mathbb{R}^{2n+2}$ is identified with the total space of $\mathcal{S}(\mathbb{R}^{2n+1},\alpha)$. This is done with the diffeomorphism $e : \mathbb{R}(t) \longrightarrow \mathbb{R}^+(t)$, $e(t) = e^t$. The use of $t \in \mathbb{R}^+$ is more convenient since we consider t to be a radius in certain polar coordinates of an annulus. The coordinate $e^t \in \mathbb{R}^+$ shall sometimes be used, as in the following example.

EXAMPLE 2.2. Consider $\mathbb{R}^{2n+1}$ with coordinates $(x_1, y_1, \ldots, x_n, y_n, z) = (\rho_1, \varphi_1, \ldots, \rho_n, \varphi_n, z)$ and endowed with the contact form

$$\alpha_0 = dz + \sum_{i=1}^{n} \rho_i^2 d\varphi_i.$$

Its symplectization is the symplectic manifold $(\mathbb{R}^{2n+1} \times \mathbb{R}(t), d(e^t\alpha_0))$. This is symplectomorphic to the standard symplectic $(\mathbb{R}^{2n+2}, \omega_0)$ where $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i + dt \wedge dz$. Indeed, consider the contact form $\widetilde{\alpha}_0 = dz - \sum_{i=1}^n y_i \cdot dx_i$ on $\mathbb{R}^{2n+1}$. It is readily seen that $(\mathbb{R}^{2n+1}, \ker \alpha_0) \cong (\mathbb{R}^{2n+1}, \ker \widetilde{\alpha}_0)$. Then the diffeomorphism

$$f : \mathbb{R}^{2n+2} \longrightarrow \mathbb{R}^{2n+2}, \quad f(x_1, y_1, \ldots, x_n, y_n, z, t) = (x_1, e^t y_1, \ldots, x_n, e^t y_n, e^t z, t)$$

pulls–back the standard symplectic form to

$$f^*\left(\sum_{i=1}^n dx_i\wedge dy_i + dt\wedge dz\right) = \sum_{i=1}^n \left(e^t dx_i\wedge dy_i + e^t y_i\cdot dx_i\wedge dt\right) + e^t dt\wedge dz =$$

$$= d\left(e^t dz - \sum_{i=1}^n e^t y_i\cdot dx_i\right) = d(e^t\widetilde{\alpha}_0).$$

Hence $\mathcal{S}(\mathbb{R}^{2n+1},\alpha_0)\cong(\mathbb{R}^{2n+2},\omega_0)$. The permutation in the variables (z,t) has its geometric origin in the dichotomy between convexity and concavity. Confer Section 2.4.

Remark 2.3. The contact structure $\xi_0=\ker\alpha_0$ on $\mathbb{R}^{2n+1}$ extends to a contact structure $(\mathbb{S}^{2n+1},\xi_0)$ via the one point compactification.

It is a natural question whether $\mathcal{S}(\mathbb{R}^3,\alpha_0)$ and $\mathcal{S}(\mathbb{R}^3,\alpha_1)$ are symplectomorphic. A symplectic topology proof could be finding exact Lagrangian tori in $\mathcal{S}(\mathbb{R}^3,\alpha_1)$, since these do not exist in $\mathcal{S}(\mathbb{R}^3,\alpha_0)$. Such a Lagrangian tori would also distinguish the symplectomorphism type of $\mathcal{S}(\mathbb{R}^3,\alpha_0)$ and $\mathcal{S}(\mathbb{R}^3,\zeta_k)$, $k\in\mathbb{Z}$. Instead, we shall use contact topology.

Note also that the classic symplectic invariants such as volume, width and symplectic capacities are necessarily infinite in the symplectization of a contact manifold.

2.3. Contactization. Let (V,λ) be an exact symplectic manifold with a Liouville 1–form λ.

Definition 2.4. The contactization $\mathcal{C}(V,\lambda)$ of (V,λ) is the contact manifold $(V\times\mathbb{R}(s),\lambda-ds)$.

Note that a different choice of primitive λ for the symplectic structure $d\lambda$ on V may lead to a different contact structure on $V\times\mathbb{R}$. In case there exists a function $f:V\longrightarrow\mathbb{R}$ such that $\lambda_0-\lambda_1=df$, the map

$$F:\mathcal{C}(V,\lambda_0)\longrightarrow\mathcal{C}(V,\lambda_1),\quad F(p,s)=(p,s-f(p))$$

is a strict contactomorphism. Note that for $V=\mathbb{R}^{2n}$, or more generally $H^1(V;\mathbb{R})=0$, such a potential f exists.

The coordinate $s\in\mathbb{R}$ in $V\times\mathbb{R}(s)$ can be considered to be an angle $s\in\mathbb{S}^1$. In particular, the contactization $\mathcal{C}(V,\lambda)$ can be compactified to $V\times\mathbb{S}^1(s)$. This compactification is also referred to as the contactization of (V,λ).

2.4. Contact fibration of $\mathcal{CS}(M,\xi)$ over $\mathbb{D}^2$. Let (M,ξ) be a contact manifold and α an associated contact form. The symplectization $\mathcal{S}(M,\alpha) \cong (M \times \mathbb{R}^+(t), d(t\alpha))$ is an exact symplectic manifold. Thus $\mathcal{CS}(M,\xi)$ is defined, the choice of Liouville form in this case is $\lambda = t\alpha$. The underlying smooth manifold $M \times \mathbb{R}^+(t) \times \mathbb{R}(s)$ can be compactified to $M \times \mathbb{R}^+(t) \times \mathbb{S}^1(s)$. Then the coordinates (t,s) can be considered to be polar coordinates on $\mathbb{R}^2 \setminus \{0\}$ and projection onto the latter two factors defines a smooth fibration

$$\pi : M \times \mathbb{R}^+(t) \times \mathbb{S}^1(s) \longrightarrow \mathbb{R}^2 \setminus \{0\}.$$

A smooth fibration $p : X \longrightarrow B$ is said to be contact for a codimension–1 distribution $\xi \subset TX$ if ξ restricts to a contact structure on any fibre. The map $p = \pi$ satisfies this condition for the natural contact structure on $\mathcal{CS}(M,\xi)$.

Proposition 2.5. *The smooth fibre bundle*

$$\pi : M \times \mathbb{R}^+ \times \mathbb{S}^1 \longrightarrow \mathbb{R}^2 \setminus \{0\}, \quad (p,t,s) \longmapsto (t,s).$$

is a contact fibration for $\xi = \ker\{t\alpha - ds\}$. *There exists a diffeomorphism* G *between contact fibrations such that*

$$\begin{tikzcd}
(M \times \mathbb{R}^+ \times (0,2\pi), \ker\{\alpha + r^2 d\theta\}) \arrow[r, "G"] \arrow[d, "p"'] & (M \times \mathbb{R}^+ \times \mathbb{R}, \ker\{t\alpha - ds\}) \arrow[d, "p"'] \\
\mathbb{R}^+ \times (0,2\pi) \arrow[r, "\cong"] & \mathbb{R}^+ \times \mathbb{R}
\end{tikzcd}$$

is commutative, the map p *being in both cases the projection onto the rightmost two factors.*

Proof. The first statement is readily verified. For the second statement, consider the following change of coordinates

$$(r,\theta) \in \mathbb{R}^+ \times (0,2\pi) \xrightarrow{g} (t,s) \in \mathbb{R}^+ \times \mathbb{R},$$
$$1/t = -4\cos^2(\theta/4) \cdot r^2, \quad s = \tan(\theta/4).$$

The map g defines a contactomorphism

$$G : (M \times \mathbb{R}^+ \times (0,2\pi), \ker\{\alpha + r^2 d\theta\}) \xrightarrow{(id,g)} (M \times \mathbb{R}^+ \times \mathbb{R}, \ker\{t\alpha - ds\})$$

since $G^*(\alpha - (1/t) \cdot ds) = \alpha + r^2 d\theta$. The map G commutes with the projections. $\square$

From the viewpoint of differential topology the projection π from $M \times \mathbb{R}^+ \times \mathbb{S}^1$ is appropriate. Nevertheless from a symplectic perspective the two ends $M_- = M \times \{0\} \times \{s_0\}$ and $M_+ = M \times \{\infty\} \times \{s_0\}$ are quite different, for any fixed $s_0 \in \mathbb{S}^1(s)$. The negative end M_- of a symplectization is concave and the positive end M_+ is convex. Consider polar coordinates $(r,\theta) \in \mathbb{R}^2$ restricting to

$$(r,\theta) \in \mathbb{R}^+ \times (0,2\pi) = \mathbb{R}^2 \setminus L,$$

where $L = \{(r,\theta) : r \geq 0, \quad \theta = 0\}$. Then the convexity of the boundary at infinity leads to the change of coordinates in Proposition 2.5. This is a more natural symplectic coordinate system: the binding of the natural open book in $\mathcal{CS}(M,\xi)$ induced by polar coordinates on the disk $\mathbb{D}^2(r,\theta)$ lies above the origin of the disk. It is then natural to compactify not only smoothly, but in a contact sense, the contact manifold $(M \times \mathbb{R}^+ \times (0,2\pi), \ker\{\alpha + r^2 d\theta\})$ to the contact manifold $(M \times \mathbb{D}^2, \ker\{\alpha + r^2 d\theta\})$.

3. Overtwisted disks and GPS

The concepts and results of this Section are part of the content of [**106, 107**].

Definition 3.1. Let (M^5,ξ) be a contact 5–fold and $\xi = \ker\alpha$. A GPS–structure is an immersion $\iota : \mathbb{S}^1 \times \mathbb{D}^2(r,\theta) \longrightarrow M$ conforming the following properties

- $\iota^*\alpha = f(r)d\theta$, for $f \geq 0$ and $f(r) = 0$ only at $r = 0, 1$.
- There exists $\varepsilon > 0$ such that the self–intersection points are of the form
$$p_1 = (s_1, r_1, \theta) \text{ and } p_2 = (s_2, r_2, \theta), \quad r_1, r_2 \in (\varepsilon, 1-\varepsilon).$$
- There exists an open set with no self–intersection points joining $\mathbb{S}^1 \times \{0\}$ and $\mathbb{S}^1 \times \partial\mathbb{D}^2$.

The existence of a GPS–structure partially restricts the fillability properties of the contact manifold $(M, \ker\alpha)$. In particular, we can use the main result in [**107**]. It implies the following

Theorem 3.1. *Let $(M, \ker\alpha)$ be a contact manifold with a GPS–structure. Then $(M, \ker\alpha)$ does not admit an exact symplectic filling.*

The construction of a GPS–structure through the use of a contact fibration was introduced in [**113**]. In Section 4 of [**107**] details for the following result are provided.

Proposition 3.2. *Let $(\mathbb{R}^3, \ker\alpha_{ot})$ be an overtwisted contact structure and $(p, r, \theta) \in \mathbb{R}^3 \times \mathbb{D}^2(r_0)$ polar coordinates. There exists $R \in \mathbb{R}^+$*

sufficiently large such that the contact manifold

$$(\mathbb{R}^3 \times \mathbb{D}^2(R), \ker\{\alpha_{ot} + r^2 d\theta\})$$

contains a GPS–structure.

4. Symplectization of an overtwisted structure

In this section we prove Theorem 1.1. Let $(\mathbb{R}^3, \ker \alpha_{ot})$ be an overtwisted contact structure. The idea is simple: the contactization $\mathcal{C}(\mathbb{R}^4, e^t\alpha_{ot})$ of the exact symplectic manifold $(\mathbb{R}^4, d(e^t\alpha_{ot}))$ is not contactomorphic to $(\mathbb{R}^5, \xi_0) \cong \mathcal{CS}(\mathbb{R}^3, \xi_0)$. Indeed, it will be proven that $\mathcal{C}(\mathbb{R}^4, e^t\alpha_{ot})$ does not embed into $(\mathbb{S}^5, \xi_0)$ whereas $(\mathbb{R}^5, \xi_0)$ does. The geometric model is that of Subsection 2.4 and thus $(\mathbb{R}^4, e^t\alpha_{ot})$ is seen as $(\mathbb{R}^3 \times \mathbb{R}^+, t\alpha_{ot})$.

LEMMA 4.1. *Let $(p, r, \theta) \in \mathbb{R}^3 \times \mathbb{R}^2$ be polar coordinates and $L = \{(p, r, \theta) : r \geq 0, \theta = 0\}$. There exists a contactomorphism*

$$\Phi : (\mathbb{R}^3 \times (\mathbb{R}^2 \setminus L), \ker\{\alpha_{ot} + r^2 d\theta\}) \longrightarrow \mathcal{C}(\mathbb{R}^3 \times \mathbb{R}^+, t\alpha_{ot}).$$

PROOF. Consider the map G in the proof of Proposition 2.5. The contactization $\mathcal{C}(\mathbb{R}^3 \times \mathbb{R}^+, t\alpha_{ot})$ is contactomorphic to

$$\mathcal{C}(\mathbb{R}^3 \times \mathbb{R}^+, t\alpha_{ot}) = \left(\mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{R}(s), \ker\{\alpha_{ot} - (1/t)ds\}\right) \xrightarrow{G^{-1}}$$

$$\xrightarrow{G^{-1}} (\mathbb{R}^3 \times \mathbb{R}^+ \times (0, 2\pi), \ker\{\alpha_{ot} + r^2 d\theta\})$$

which is $(\mathbb{R}^3 \times (\mathbb{R}^2 \setminus L), \ker\{\alpha_{ot} + r^2 d\theta\})$. □

LEMMA 4.2. *Let $(p, x, y) = (p, r, \theta) \in \mathbb{R}^3 \times \mathbb{R}^2$ be cartesian and polar coordinates. There exists a strict contactomorphism*

$$\Psi : (\mathbb{R}^3 \times \mathbb{R}^2, \ker\{\alpha_{ot} + r^2 d\theta\}) \longrightarrow (\mathbb{R}^3 \times \mathbb{R}^2, \ker\{\alpha_{ot} - y dx\}).$$

which preserves the fibres of the projection onto the second factor.

PROOF. The contact form $\alpha_{ot} + r^2 d\theta$ in Cartesian coordinates reads $\beta_0 = \alpha_{ot} + \frac{1}{2}(x dy - y dx)$. Consider the homotopy of contact forms

$$\beta_t = \alpha_{ot} - y dx + \frac{1-t}{2}(x dy + y dx), \quad t \in [0, 1].$$

It begins at β_0 and ends at $\beta_1 = \alpha_{ot} - y dx$. Let us find an isotopy τ_t solving the equation $\tau_t^* \beta_t = 0$. Suppose that τ_t is the t–time flow of a vector field X_t. The derivative of the equation reads

$$\tau_t^*(\mathcal{L}_{X_t}\beta_t + \dot{\beta}_t) = 0, \text{ i.e. } d\iota_{X_t}\alpha_t + \iota_{X_t} d\alpha_t - \frac{1}{2}(x dy + y dx) = 0.$$

Note that $d(xy) = xdy + ydx$ and thus the autonomous vector field

$$X = \frac{xy}{2} R_{ot}$$

is the solution to this equation, where R_{ot} denotes the Reeb vector field of α_{ot}. The vector field X is a complete vector field in $\mathbb{R}^3 \times \mathbb{R}^2$. Let τ be its 1–time flow. Then the diffeomorphism

$$\Psi : \mathbb{R}^3 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \times \mathbb{R}^2, \quad \Psi(p, x, y) = (\tau(p, x, y), x, y)$$

satisfies $\Psi^*(\alpha_{ot} - ydx) = \alpha_{ot} + r^2 d\theta$. □

THEOREM 4.1. *There exists no contact embedding* $\mathcal{CS}(\mathbb{R}^3, \alpha_{ot}) \longrightarrow \mathcal{CS}(\mathbb{R}^3, \alpha_0)$.

PROOF. The contact manifold $\mathcal{CS}(\mathbb{R}^3, \alpha_0)$ is contactomorphic to $(\mathbb{R}^5, \ker \alpha_0)$. Thus it embeds via the inclusion j into $(\mathbb{S}^5, \ker \alpha_0)$. The contact manifold $(\mathbb{S}^5, \ker \alpha_0)$ admits an exact symplectic filling by the standard symplectic ball $(\mathbb{D}^6, \omega_0|_{\mathbb{D}^6})$. Suppose that there exists a contact embedding

$$h : \mathcal{CS}(\mathbb{R}^3, \alpha_{ot}) \longrightarrow \mathcal{CS}(\mathbb{R}^3, \alpha_0).$$

Proposition 3.2 implies the existence of a GPS–structure on the contact manifold

$$\mathcal{N}_R = (\mathbb{R}^3 \times \mathbb{D}^2(R), \ker\{\alpha_{ot} + r^2 d\theta\})$$

for R large enough. Let us show that $\mathcal{N}_R$ contact embeds into $\mathcal{CS}(\mathbb{R}^3, \alpha_{ot})$. Lemma 4.1 identifies this contactization via Φ with $(\mathbb{R}^3 \times (\mathbb{R}^2 \setminus L), \ker\{\alpha_{ot} + r^2 d\theta\})$. The contactomorphism Ψ in Lemma 4.2 allows us to use $(\mathbb{R}^3 \times (\mathbb{R}^2 \setminus L), \ker\{\alpha_{ot} - ydx\})$.

Consider an arbitrary $R_0 \in \mathbb{R}^+$, the inclusion $i : \mathbb{D}^2(R_0) \longrightarrow \mathbb{R}^2(x, y)$ as a disk centered at the origin and the diffeomorphism $f_R \in \mathrm{Diff}(\mathbb{R}^2)$ defined as $f_R(x, y) = (x - R, y)$. The image of $\mathcal{N}_{R_0}$ via the contact embedding $(id, f_R \circ i)$ is contained in $\mathbb{R}^3 \times (\mathbb{R}^2 \setminus L)$ if $R > R_0$. It is readily verified that

$$\gamma : \Phi \circ \Psi^{-1} \circ (id, f_{2R_0} \circ i) : \mathcal{N}_{R_0} \longrightarrow \mathcal{CS}(\mathbb{R}^3, \alpha_{ot})$$

is a contact embedding. The radius R_0 can be chosen arbitrarily large. The map $j \circ h \circ \gamma$ endows $(\mathbb{S}^5, \ker \alpha_0)$ with a GPS–structure. This contradicts Theorem 3.1. □

Proof of Theorem 1.1: Suppose that symplectic structure $\mathcal{S}(\mathbb{R}^3, \alpha_{ot})$ is not exotic. Then there exists an embedding $i : \mathcal{S}(\mathbb{R}^3, \alpha_{ot}) \longrightarrow \mathcal{S}(\mathbb{R}^3, \alpha_0)$. It induces a contact embedding

$$j : \mathcal{CS}(\mathbb{R}^3, \alpha_{ot}) \longrightarrow \mathcal{CS}(\mathbb{R}^3, \alpha_0).$$

This contradicts Theorem 4.1. □

Note that the symplectic structure $\mathcal{S}(\mathbb{R}^3, \xi_{ot})$ is never standard at infinity. It has been proven by M. Gromov that a symplectic structure on $\mathbb{R}^4$ standard at infinity is necessarily isomorphic to the standard symplectic structure $(\mathbb{R}^4, \omega_0)$.

The contact structures ξ_0 and ξ_1 on $\mathbb{R}^3$ are homotopic through contact structures. This homotopy can be obtained by dilating the overtwisted disks off to infinity. This geometric path of contact structures yields a path of exact symplectic forms joining the standard symplectic structure ω_0 and the symplectic structure on $\mathcal{S}(\mathbb{R}^3, \xi_1)$. A visual homotopy between ξ_0 and ζ_k can be readily constructed using contractions to a Darboux ball. This also induces a homotopy between ω_0 and the symplectic form of $\mathcal{S}(\mathbb{R}^3, \zeta_k)$.

5. Examples of Non–Isomorphic Symplectizations

In this Section we provide details on Theorem 3 and Theorem 1.3.

5.1. Open contact 3–folds. In Section 3 we have shown that $\mathcal{S}(\mathbb{R}^3, \alpha_0)$ is not symplectomorphic to $\mathcal{S}(\mathbb{R}^3, \alpha_{ot})$. The procedure we used yields several examples of open manifolds exhibiting this behaviour. In particular Theorem 3 stated in the introduction.

THEOREM 3. *Let (M, ξ) be an exact symplectically fillable contact manifold and $(U, \xi) \subset (M, \xi)$ an open contact submanifold. Consider an overtwisted contact structure (U, ξ_{ot}). Then $\mathcal{S}(U, \xi)$ is not symplectomorphic to $\mathcal{S}(U, \xi_{ot})$.*

PROOF. Consider an exact symplectic filling (W, λ) for (M, ξ), $\xi = \ker \alpha$. Note that $\mathcal{S}(M, \xi)$ embeds into (W, λ) as a neighborhood of the boundary. The contact 5–fold $\mathcal{C}(W, \lambda) = (W \times \mathbb{S}^1, \lambda - ds)$ has boundary $M \times \mathbb{S}^1$. In order to obtain a closed contact 5–fold (X, Ξ) we glue the manifold $(M \times \mathbb{D}^2, \alpha + \rho^2 d\varphi)$ along their common boundary $M \times \mathbb{S}^1$. The manifold (X, Ξ) admits an exact symplectic filling.

Observe that the open contact manifold (U, ξ) embeds into (X, Ξ) with an arbitrarily large neighborhood. Indeed, (M, ξ) has an arbitrarily large symplectic neighborhood in (W, λ). For instance, it can be obtained by expanding a given neighborhood with the Liouville flow.

The open contact manifold $\mathcal{CS}(U,\xi_{ot})$ contains a GPS–structure. Suppose that $\mathcal{S}(U,\xi)$ is symplectomorphic to $\mathcal{S}(U,\xi_{ot})$, then $\mathcal{S}(U,\xi_{ot})$ embeds into (W,λ). Hence the contact manifold $\mathcal{CS}(U,\xi_{ot})$ embeds into (X,Ξ). This contradicts Theorem 3.1. □

REMARK 5.1. The manifold (X,Ξ) used in the proof is not unique. The relative suspension using a composition of positive Dehn twists also yields an exact symplectically fillable manifold and the argument applies.

5.2. Higher Dimensions. Consider an overtwisted contact structure $(\mathbb{R}^3,\xi_{ot})$ and polar coordinates $(\rho_1,\varphi_1,\ldots,\rho_{n-2},\varphi_{n-2})\in\mathbb{R}^{2n-4}$. The contact structure ξ_{ex} defined by the kernel of the 1–form

$$\alpha_{ex}=\alpha_{ot}+\sum_{i=1}^{n-2}\rho_i^2 d\varphi_i$$

contains a GPS–structure. Thus it is not contactomorphic to $(\mathbb{R}^{2n-1},\xi_0)$. The statement of Proposition 3.2 also holds for the contact manifold $(\mathbb{R}^{2n-1},\xi_{ex})$. That is, there exists a GPS–structure on $(\mathbb{R}^{2n-1}\times\mathbb{D}^2(R),\alpha_{ex}+r^2d\theta)$. Confer [**107**] for details. The existence of this GPS–structure and the analogues of Lemmas 4.1 and 4.2 prove that $\mathcal{CS}(\mathbb{R}^{2n-1},\alpha_{ex})$ does not contact embed into $\mathcal{CS}(\mathbb{R}^{2n-1},\alpha_0)$. The same argument used in Theorem 1.1 yields the following

PROPOSITION 5.2. *Let $(\mathbb{R}^3,\xi_{ot})$ be an overtwisted contact structure, $\xi_{ot}=\ker\alpha_{ot}$. Then the symplectization $\mathcal{S}(\mathbb{R}^{2n-1},\alpha_{ex})$ endows $\mathbb{R}^{2n}$ with an exotic symplectic structure.* □

This allows us to conclude Theorem 1.3 stated in the introduction.

Proof of Theorem 1.3: Consider the diffeomorphism

$$f:\mathbb{R}^{2n}\longrightarrow\mathbb{R}^{2n},\quad f(\rho,\varphi,z,t;\rho_1,\varphi_1,\ldots,\rho_{n-2},\varphi_{n-2})=$$
$$=(\rho,\varphi,z,t;e^{t/2}\rho_1,\varphi_1,\ldots,e^{t/2}\rho_{n-2},\varphi_{n-2}).$$

Consider the 1–forms

$$\widetilde{\lambda}_{ex}=e^t\alpha_{ot}+\sum_{i=1}^{n-2}\rho_i^2 d\varphi_i,\quad \lambda_{ex}=e^t\alpha_{ex}=e^t(\alpha_{ot}+\sum_{i=1}^{n-2}\rho_i^2 d\varphi_i)$$

The diffeomorphism f pulls–back $f^*\widetilde{\lambda}_{ex} = \lambda_{ex}$. In particular
$$(\mathcal{S}(\mathbb{R}^3, \xi_{ot}) \times \mathbb{R}^{2n-4}, \omega_{ot} + \omega_0) \cong \mathcal{S}(\mathbb{R}^{2n-1}, \alpha_{ex})$$
are symplectomorphic. This concludes the statement. $\square$

Proposition 5.2 can also be used to prove an analogue of Theorem 3 in higher dimensions.

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